TEACHER CONTENT COMPETENCIES

Enabling Content and Pedagogical Understandings for Effective Content Instruction

MATHEMATICS, GRADES 3-5

DRAFT – WINTER 2020
This is a working draft to be used for piloting and feedback. This document will continue to be improved over the coming years.

TABLE OF CONTENTS

Context p.1

The Content Competency Framework p.3

Attention to Equity: Overarching Competencies p.4

The Mathematics 3-5 Content Competency Framework p.6

Multiplication and Division: Building Procedural Fluency from Conceptual Understanding p.7

- Properties of Operations & Fact Fluency p.7
- Conceptual Foundations of Multi-Digit Multiplication p.18
- Conceptual Foundations of Multi-Digit Division p.35

Fractions: Connecting Visual Representations to Abstract Reasoning p.43

- Fractions as Numbers p.43
- Equivalent Fractions p.51
- Fraction Addition and Subtraction p.60
- Fraction Multiplication and Division p.70

Deconstructing Word Problems: Promoting Access to Language and Mathematical Concepts for All Learners p.80

- Addition and Subtraction Problem Types with Supporting Visuals p.82
- Multiplication and Division Problem Types with Supporting Visuals p.97
- Multi-Step Problems p.107

Curriculum Understanding: Overarching Competencies p.113

Works Cited p.115

Appendix p.121

Context

Every child is capable of being a "math person." Every child deserves to feel confident in their mathematics skills and knowledge and tackle complex and real world challenges with ease and excitement. Occupations that require fluency with mathematics are growing at 17%, while other occupations are growing at 9.8% (U.S. Bureau of Labor Statistics). Every child is capable of contributing to math fields and deserves the preparation that positions them to do so if they choose.

Great teaching is a critical part of the path to equity and opportunity. We know from Richard Elmore, "There are only three ways to improve student learning at scale: You can raise the level of the content that students are taught. You can increase the skill and knowledge that teachers bring to the teaching of that content. And you can increase the level of students' active learning of the content. That's it." (Elmore) While teaching is complex work, it can be taught. We can and must systematically prepare and support teachers with the knowledge, skills, and mindset they need to consistently enact effective and equitable instruction that leads all students to learn and love mathematics.

Teachers work hard and want to learn and leverage whatever will help their students most. Teachers work, on average, 53 hours per week (Scholastic, 2014). 79% of teachers want better training on the standards to teach their students more effectively (TNTP, 2018). Teaching is highly demanding intellectual and human work and teachers deserve the very best preparation and ongoing support. We believe it is possible to make the job of teaching and supporting teachers slightly easier and to make teaching more consistently excellent for all students.

Although driven by good intentions, current teacher preparation and professional learning efforts are based on competing frameworks about what matters and, too often, inattention to the critical details of teaching. This lack of clarity and coherence leaves teachers to independently determine what matters most in their classroom for their students. Furthermore, professional learning efforts rarely lead to improved practice or more student learning. While districts spend an average of $18,000 per teacher per year on training, seven out of ten of those teachers remain the same or decline in their evaluation ratings (TNTP, 2015). This suggests that for all of the money spent to help teachers improve, very little of it translates to improved teaching quality and growth in student learning.

High-quality instructional materials play an important role in supporting students and teachers. A strong curriculum has a positive impact on student learning and helps lessen the burden for teachers to independently create and source materials, allowing them to focus their energy on engaging students. A growing number of schools are seeking high-quality instructional materials as the foundation for equitable instruction. However, materials alone do not transform the quality of teaching. Teachers continue to rely on their mental models of what good teaching looks like (often heavily informed by how they were personally taught) as the prism through which they filter their materials. We have observed that teachers' capacity (their knowledge, skill, and mindset about their content and how to teach their content) governs their approach to instruction, even when supported by high-quality instructional materials. Interacting with high-quality instructional materials can change teacher capacity (the materials can be educative and instructive) and play an important role in giving teachers new ideas about what great teaching can look like and how to approach key content. However, it is the capacity, not the materials, that guide the thousands of decisions teachers need to make each lesson. Therefore, it is no surprise that effective professional learning focused on high-quality curricular materials is the combo power-play that best helps teachers improve (Hill).

Different organizations (districts, publishers, and professional learning providers) take different approaches in focus (content and objectives) and in model (dosage, delivery method, and context) of curriculum-based professional learning. At Instruction Partners, we are a nonprofit organization that partners with small school systems to improve teaching and accelerate learning. We do this through continuous improvement observation routines with school and school system leaders, academic strategy consulting, and capacity building for teachers and leaders through professional learning, usually grounded in high-quality instructional materials. To better understand our approach relative to others', we interviewed professional learning organizations, curriculum developers, districts/charter management organizations, and researchers and reviewed teacher learning practices from other countries. We sought to better understand the specific competencies teachers need to possess to teach their content effectively so we could strengthen our professional learning with partners to lead to more effective instruction and learning. This led us to first seek and, when we could not find a framework with sufficient detail, to develop the Teacher Content Competencies.

The Mathematics Teacher Content Competencies seek to outline comprehensively and specifically the researched-based knowledge, skills, and mindsets that 3-5 grade math teachers need, in partnership with a high-quality curriculum, in order to teach all students effectively. Just as the standards for student learning create a clear understanding of the outcomes students need to be ready for college and career, the Teacher Content Competencies seek to detail the teacher learning outcomes necessary for success. And, just as standards for student learning become the backbone for curriculum and assessment, we hope the Teacher Content Competencies can become the backbone for more coherent and clear educator professional learning and assessment in our own organization. Finally, like the standards, the competencies are also rooted in a deep research base. Throughout the document you will see citations followed by a complete works cited for all elements of the competencies.

This is a draft that we intend to pilot, seek feedback, and iterate over the coming year. We share this draft under the CC-BY-SA license with others so that others can use the competencies, test them, and share reflections as we collectively work to improve our field. We license this "share-alike" because, as it evolves, we do not want to create competing versions but seek to evolve it in community. This is the first complete draft to be published. While this document is a complete set of the math competencies, it remains a draft and we will update this draft over time, frequently in the next couple of years. As you review these competencies or attempt to use them, we welcome your feedback. **Please submit any notes or recommendations at bit.ly/competenciesfeedback.**

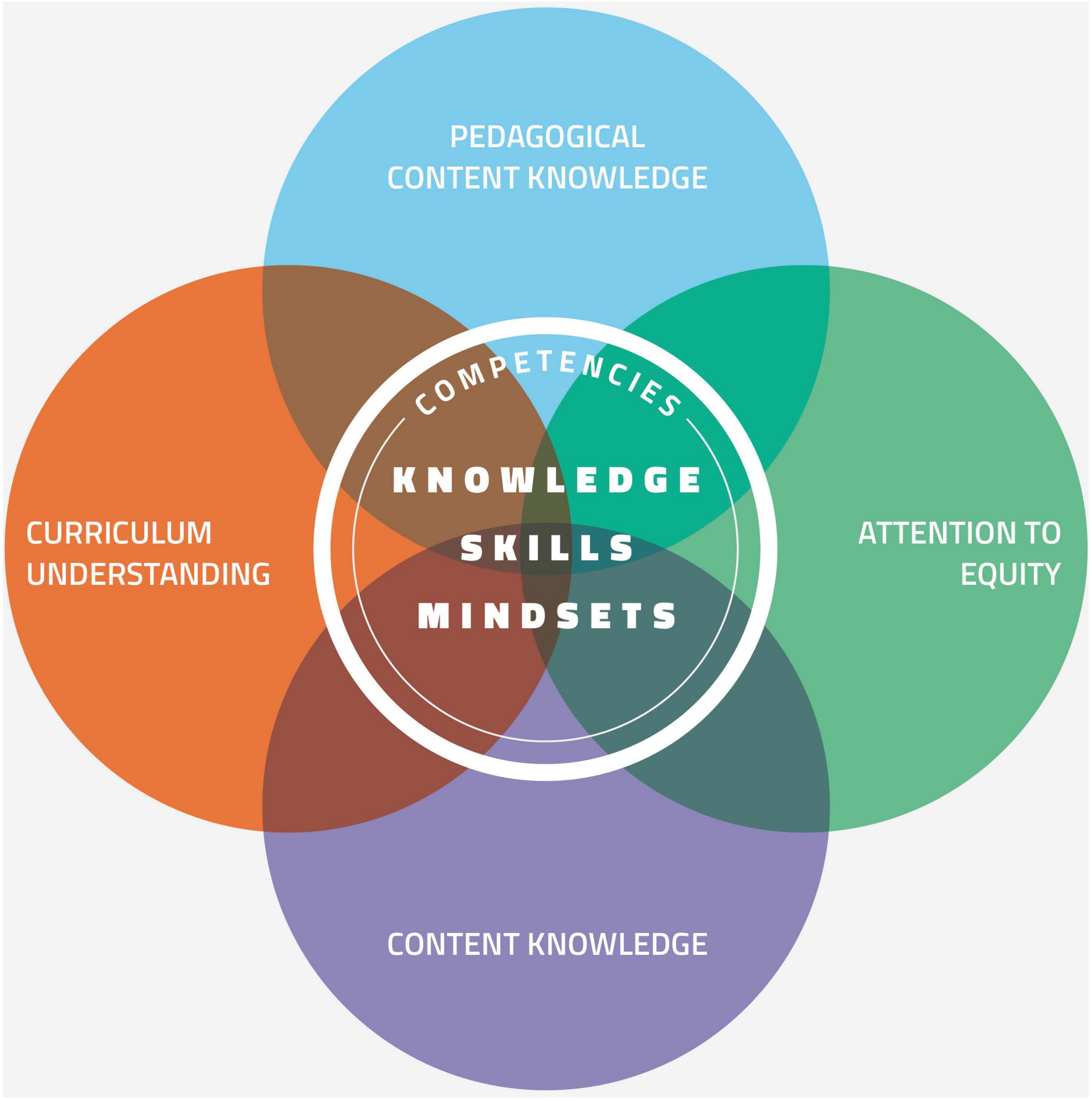
The Content Competency Framework

The Teacher Content Competencies explicitly detail the elements necessary for excellent math instruction. In any work, excellence in action is enabled by a set of knowledge, skills, and mindsets that guide actions. These competencies represent the knowledge, skills, and mindsets necessary for great grade 3-5 mathematics instruction.

These competencies are not intended to represent the entire realm of capacity to support great teaching. Rather, they specifically hone in on four primary areas of instruction:

- Attention to Equity
- Curriculum Understanding
- Content Knowledge (3-5 mathematics)
- Pedagogical Content Knowledge (3-5 mathematics)

We treated each area uniquely and thoroughly. The areas of Attention to Equity and Curriculum Understanding are detailed in two different ways. First, they are detailed individually. These sections include the general knowledge, skills, and mindsets all teachers need, no matter their content area, to succeed. Second, they are detailed within the mathematics content knowledge and pedagogical content knowledge competencies.



Attention to Equity: Overarching Competencies (AE)

Great instruction relies on a large set of underlying beliefs and skills to ensure success. While it is right to look to the specific content knowledge, pedagogical content knowledge, and classroom management skills needed for success, that view alone is incomplete. Great instruction also flows from a teacher's unwavering belief in the potential of every student they serve and concrete skills that allow them to support and educate every student, including those from different backgrounds than their own. We know students in poverty, students of color, students with disabilities, and students learning English are enrolled in every school and district in this country. Research shows that, when given access to rigorous tasks and aligned instructional experiences, students in all identity groups achieve at a high level (TNTP, 2018). This section outlines the overarching competencies necessary for every teacher, no matter the content they teach. We know that student equity lives in the daily decisions a teacher makes in the context of the content being taught. The skills needed to make those daily decisions are addressed in the competencies as well.

Attention to Equity: Beliefs about Children and Their Potential (AE.BP)

AE.BP.1: Have high-regard for and believe in the academic potential and the deservedness or worthiness of students and their communities.
Supporting research: Teach For America, 2013; Ladson-Billings (2009); Love & Kruger (2005); Irvine (2003) ELL-specific: de Jong, & Harper (2005); Walker, Shafer, & Iiams (2004); Irvine (2003)

AE.BP.2: Believe in the importance of seeing, naming, and welcoming the multiple identities of the children, rather than profess any sort of "color-blindness" that acts to deny the life experiences of marginalized people and to compound inequities.
Supporting research: Bell (2002); Ullici and Battey (2011); Hachfield, Hahn, Schroeder, Anders, & Kunter (2015); Gay (2000); Delpit (1995)

AE.BP.3: Hold high expectations for the effort, persistence, and the achievement of all students.
Supporting research: TNTP (2018); McKown & Weinstein (2008) ELL-specific: de Jong & Harper (2005) McGrady & Reynolds 2012; Rist (1970); Jussim & Harber, (2005); Hinnant, O'Brien, & Ghazarian (2009); McKown & Weinstein (2008); Wentzel (2002); NCTM (2014)

Attention to Equity: Beliefs about the Centrality of Relationships (AE.BR)

AE.BR.1: Believe in the importance of building trusting relationships with students through enacting beliefs about children and their potential and beliefs about the inequity in schools.
Supporting research: Yeager, Purdie-Vaughns, Garcia, Apfel, Brzustoski, Master, Hessert, Williams, & Cohen (2013); Irvine (2003)

AE.BR.2: Believe in the importance of building trusting relationships with students by seeking out and understanding the similarities between teachers and students.
Supporting research: Gehlbach, Brinkworth, King, Hsu, McIntyre, & Rogers (2015)

AE.BR.3: Believe in the importance of building trusting relationships with students by cultivating a sense of collective community in the classroom and/or by being in the community with them. *Supporting research: Ladson-Billings (2009); Love & Kruger (2005); Irvine (2003)*

Attention to Equity: Beliefs about Culture (AE.BC)

AE.BC.1: Believe that classrooms and schools have cultures (e.g., accepted ways of operating and being), that everyone who comes to school comes from a unique home culture, none of which is inherently superior to another, and that for learning and well-being to be maximized, each student's home culture needs to be welcomed and used as a vehicle for learning. *Supporting research: Irvine (1990); Irvine (2003); Ladson-Billings (1995); Ladson-Billings (2009); Stigler & Hiebert (2009); Gay (2000); Delpit (1995) ELL-specific: Walker, Shafer, & Iiams (2004); Yoon (2008)*

AE.BC.2: Believe that for all children to be successful, the cultural norms for "doing school" (generally) or the discipline (specifically) need to be made visible and taught explicitly (e.g., in mathematics, what counts as acceptably explaining your work). *Supporting research: Delpit (1998) Math-specific: Yackel and Cobb (1996); Boaler (2002)*

Attention to Equity: Beliefs about the Teacher and the Teacher's Potential for Impact (AE.BT)

AE.BT.1: Believe in the importance of examining a teacher's personal worldview and what has shaped it, including their upbringings, their multiple identities, and their socio-cultural location, in order to surface personal biases, areas of unearned advantage and/or disadvantage, blind spots, and power and examine how effectively those are used to pursue equity. *Supporting research: Irvine (2003); Kailin (1994); Kailin (1999); King and Ladson-Billings (1990); Ullici and Battey (2011)*

AE.BT.2: Hold empowered beliefs, internal locus of control, and take responsibility for students' learning.
Supporting research: Rochmes (2015); Ladson-Billings (2009); Lee & Smith (1996); Irvine (2003) ELL-specific research: Walker, Shafer, & Iiams (2004); Yoon (2008)

AE.BT.3: Hold self-efficacious beliefs (i.e., believe that all students can be successfully taught content, even those typically viewed as difficult).
Supporting research: (Beilock, Gunderson, Ramirez, and Levine (2010); Caprara, Barbaranelli, Steca, & Malone (2006); Holzberger & Kunter (2013); Tschannen-Moran & Woolfolk Hoy (2001) ELL-specific: Karabenick & Noda (2004)

Attention to Equity: Beliefs About Learning and the Content or Discipline One Teaches (AE.BL)

AE.BL.1: Believe that learning (broadly) and the content (specifically) must be relevant to students' present lives, future dreams, and/or goals to improve the community and world.
Supporting research: Hulleman & Harackiewicz (2009); Yeager, Purdie-Vaughns, Garcia, Apfel, Brzustoski, Master, Hessert, Williams, & Cohen (2014); Ladson-Billings (2009); Gonzalez, Moll, & Amanti (2005) Math-specific research: Boaler & Staples, 2008; Davis, West, Greeno, Gresalfi, & Martin, (2007); Moses & Cobb (2002); Civil (2007)

AE.BL.2: Believe that students must be able to affiliate with—and envision themselves as a productive doer of—the discipline.
Supporting research: Nasir, Shah, Gutierrez, Seashore, Louie & Baldinger (2011); Martin, (2000); de Abreu, (1995); Sfard & Prusak, (2005); Moody (2004); de Abreu & Cline, (2003); Louie (2017)

The Mathematics 3-5 Content Competency Framework

The Mathematics Teacher Content Competencies illustrate a vision for how students learn elementary mathematics, the foundations for more complex mathematics.

Specifically, the math competencies focus a teacher on the four critical areas a teacher must answer to be prepared to teach effectively.

1. *Why Does the Math Makes Sense?* Teachers must understand the progression of mathematical concepts and why the math they teach makes sense. The content in this section requires content knowledge beyond what is asked of students in the standards.

2. *How is This Concept Connected to Other Concepts?* Teachers must understand the content of a given unit as part of a larger coherent structure within and across grades. Teachers must understand that the work students do with area models to multiply whole numbers will eventually expand to include multiplying fractions; the conceptual foundations are the same and should be leveraged as such.

3. *What Must I Understand About Teaching This Content?* The content in this section provides guidance on the types of representations, tasks, and instructional competencies that would be helpful in teaching specific content. This section also sheds light on common instructional pitfalls, such as teaching keywords, which should be explicitly named and discussed.

4. *What Must I Understand About Student Reasoning in This Content?* The content in this section addresses common patterns of student thought around given content and also offers guidance for instructional support that would benefit students with learning differences.

The framework helps a teacher leverage the three most critical content progressions for this grade band, Multiplication and Division, Fractions, and Deconstructing Word Problems in the context of those core questions to prepare for instruction. This integration helps teachers go beyond simply understanding the specific content they are going to teach, but rather learn the content in a way that prepares them to effectively teach it within the context of using a high-quality curriculum.

In order for teachers to apply their learnings to their curriculum, each content competency contains curriculum application guidance where teachers examine the design of their curriculum around priority content and concepts and then look for specific tasks that either illuminate key understandings or would be ideal for surfacing and addressing specific misconceptions. Though there is guidance in each content focus area narrative around recommended instructional priorities, the method of instructing is intentionally left open to accommodate different curricular or school-specific practices and routines. Ideally, the learning within each content focus area would be geared toward one or two long-term instructional priorities (such as the implementing the Practices or using formative assessment to understand and respond to student thinking).

Content Focus Area:

Multiplication and Division: Building Procedural Fluency from Conceptual Understanding

The following competencies within this content focus area aim to ensure teachers are prepared to foster procedural fluency in multiplication and division defined as:

1. Accuracy: the ability to produce mathematically precise answers
2. Efficiency: the ability to produce answers relatively quickly and easily
3. Appropriate strategy use: the ability to select and apply a strategy that is appropriate for solving the given problem efficiently
4. Flexibility: the ability to think about a problem in more than one way and to adapt or adjust thinking if necessary (National Research Council, 2001, p. 116)

In order to promote procedural fluency, teachers must understand the student learning progression and ensure students have strong conceptual understanding that helps them connect new ideas to existing knowledge.

Properties of Operations & Fact Fluency (PO)

This competency describes how teachers support students' transition into multiplication and division. It focuses on decontextualized math — how students learn to operate flexibly, efficiently, and accurately with multiplying and dividing numbers. These strategies co-evolve with student understanding of contextualized problems (understanding partitive and measurement division, multiplicative comparison, etc.) which are introduced here but explored in depth within the Deconstructing Word Problems: Promoting Access to Language and Mathematical Concepts for All Learners section.

This content competency begins by developing the foundations for the following arithmetic properties involving multiplication: the commutative property of multiplication, the associative property of multiplication, and the distributive property of multiplication over addition. Students develop the arithmetic properties as they become comfortable with the concept of multiplication and division and move toward finding efficient pathways to leverage known number relationships (such as 5 x 8) to figure out unknown relationships (such as 6 x 8). The arithmetic properties established both promote fluency with multiplication facts but also lay the foundation for multi-digit multiplication and division.

This competency heavily focuses on the content teachers must know, though the methods of teaching are intentionally left open to allow teachers to work within the guidance of their curriculum. With that in mind, there are several practices that are highly effective for this module that should be named clearly. Grade 3 teachers must create opportunities for students to discuss and compare their reasoning about multiplication facts both through written work and also mental math exercises. Teachers must also deeply understand the strategies students use to avoid the common but widely ineffective practice of "teaching fluency" through timed tests and rote memorization. Grade 4 and 5 teachers must understand that timed tests and rote memorization are equally unhelpful for students who do not have strategies to derive unknown facts. By integrating the derived fact strategies students use to become fluent and focusing on small sets of related number facts, grade 4 and 5 teachers can support students with unfinished learning. When students have reasoning strategies to derive specific multiplication facts, teachers can focus on speed and accuracy through small practice sets on related facts with feedback (National Council of Teachers of Mathematics, 2014).

Focus Standards

3.OA.5: Understand properties of multiplication and the relationship between multiplication and division. Apply properties of operations as strategies to multiply and divide. Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known (Commutative property of multiplication). 3 × 5 × 2 can be found by 3 × 5 = 15 then 15 × 2 = 30, or by 5 × 2 = 10 then 3 × 10 = 30 (Associative property of multiplication). Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56 (Distributive property). (Students need not use formal terms for these properties.)

3.OA.6: Understand properties of multiplication and the relationship between multiplication and division. Understand division as an unknown-factor problem. For example, divide 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.

3.OA.7: Multiply and divide within 100. Fluently multiply and divide within 100 using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of one-digit numbers.

Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category

3.OA.9: Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into 2 equal addends.

3.NBT.3: Use place value understanding and properties of operations to perform multi-digit arithmetic. Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations. (A range of algorithms may be used.)

3.MD.7: Geometric measurement: understand concepts of area and relate area to multiplication and to addition. Relate area to the operations of multiplication and addition.

3.MD.7a: Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

3.MD.7c: Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and b + c is the sum of a × b and a × c. Use area models to represent the distributive property in mathematical reasoning.

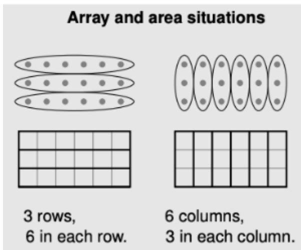
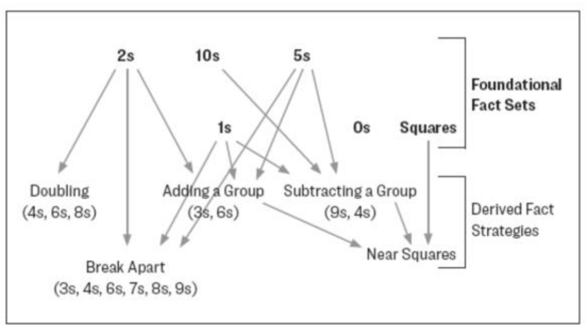
5.MD.5a: Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths equivalently by multiplying the height by the area of the base. Represent three-fold whole-number products as volumes (e.g., to represent the associative property of multiplication).

Why Does the Math Make Sense? (PO.M)	How is This Concept Connected to Other Concepts? (PO.CC)	What Must I Understand About Teaching This Content? (PO.TC)	What Must I Understand About Student Reasoning in This Content? (PO.SR)
PO.M.1: Understanding the Concept Progression **PO.M.1a:** Identify multiplication (M x N) as the number of units (or objects) in M equal groups if there are N units (or objects) in one group[1] (Beckmann, 2011, p. 143). **PO.M.1b:** Understand that whereas addition and subtraction require the same units, multiplication can refer to different units (e.g., 3 baskets with 5 apples in each basket simultaneously coordinates the units of baskets and apples) (Beckmann, 2011, p. 143). **PO.M.1c:** Construct physical or visual representations of equal groups and/or arrays and use strategies to find the total (strategies can begin with counting all, to skip counting, to more efficient strategies). **PO.M.1d:** Represent arrays and use repeated addition or multiplication to show ways to find the total numbers of objects in the array. Recognize that arrays can be split into equal numbers of rows or equal numbers of columns (Beckmann, 2011, p. 145).	**PO.CC.1: Connections to Prior Learning** **PO.CC.1a:** Connect work with equal groups multiplication to prior work with repeated addition and skip-counting in grade 2.[5] **PO.CC.1b:** Connect the roles of the associative and commutative properties of addition in supporting addition fact fluency[6] to the roles of the associative and commutative properties of multiplication in supporting multiplication fact fluency. **PO.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Module** **Grade 3:** **PO.CC.2a:** Explain the role of the associative property in mentally multiplying a whole number by a	**PO.TC.1: Visual Diagrams and Representations** **PO.TC.1a:** Recognize multiple visual representations that could be used to demonstrate M x N as the number of units in M equal groups if there are N units in each group, including: • Arrays • Area models • Equal groups drawings (such as 5 x 4 being 5 circles, each with 4 stars inside) **PO.TC.1b:** Understand the value of the array and area models, including: • They can be used to demonstrate different applications of the properties of arithmetic. • The array can be generalized as an area model, which in turn facilitates visually representing the product of two multi-digit numbers.	**PO.SR.1: Diagnosing Student Reasoning** **PO.SR.1a:** Identify the components of procedural fluency as: • Accuracy: the ability to produce mathematically precise answers • Efficiency: the ability to produce answers relatively quickly and easily • Appropriate strategy use: the ability to select and apply a strategy that is appropriate for solving the given problem efficiently • Flexibility: the ability to think about a problem in more than one way and to adapt or adjust thinking if necessary (National Research Council, 2001, p. 116) **PO.SR.1b:** Given student strategies for single digit multiplication, identify which components of procedural fluency are or are not present (e.g., skip counting by 8s

[1] This ordering convention represents the common convention in the United States, though other countries have different orderings and interpretations.

[5] 2.OA.C.4: Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends; 2.NBT.A.2: Count within 1000; skip-count by 5s, 10s, and 100s.

[6] Apply properties of operations as strategies to add and subtract. *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known (Commutative property of addition). To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12 (Associative property of addition).*

Why Does the Math Make Sense? (PO.M)	How is This Concept Connected to Other Concepts? (PO.CC)	What Must I Understand About Teaching This Content? (PO.TC)	What Must I Understand About Student Reasoning in This Content? (PO.SR)
 (CCSS Writing Team, 2019, p.33) **PO.M.1e:** Construct physical or visual representations of equal groups and/or arrays and use strategies to answer division questions of how many units in each group or how many groups. **PO.M.1f:** Identify and apply common strategies associated with foundational facts: **Figure 1.3. Multiplication Fact Fluency Flexible Learning Progression**  (Bay-Williams & Kling, 2019, p. 63)	multiple of 10.[7] **PO.CC.2b:** Identify patterns within the multiplication tables and explain them using reasoning based on the arithmetic properties.[8] **Grade 5:** **PO.CC.2f:** Explain that we can find the volume of a box, in cubic units, by finding the number of 1-unit-by-1-unit-by-1-unit cubes it takes to fill or make the box (without gaps or overlaps) (CCSS Writing Team, 2019, p. 110). **PO.CC.2g:** Explain that we can multiply H*(L*W) to find the volume of a box that is H units high, L units long, and W units wide because the box can be subdivided into H layers, each of which contains L rows of W 1-unit-by-1-unit-by-1-unit cubes (Beckmann, 2011, p. 194). **PO.CC.2h:** Explain how the associative property can be demonstrated by building rectangular prisms from unit cubes and using grouping symbols to	**PO.TC.2: Symbolic Representations** **PO.TC.2a:** Identify the appropriate notation for introducing multiplication (x) and recognize other forms such as dot notation and "next to" notation usually emerge in later grades. **PO.TC.2b:** Identify the appropriate notation for introducing division. **PO.TC.2c:** Explain how writing quantities in unit form helps students decompose quantities to apply the associative property of multiplication (e.g., 4 tens x 5 is 20 tens). **PO.TC.3: Recommended Instructional Competencies** **PO.TC.3a:** Pose a series of mental math problems and have students verbalize their reasoning. **PO.TC.3b:** Script student reasoning from verbal descriptions of mental math clearly. **PO.TC.3c:** Select and sequence student work that could highlight an important mathematical connection (such as the	to find 8 x 9 may be accurate but not efficient, appropriate, or flexible). **PO.SR.1c:** Given student work on single digit multiplication, identify the Level 1, Level 2, or Level 3 strategies used. **PO.SR.2: Student Misconceptions and Difficulties** **PO.SR.2a:** Difficulty: Simultaneous coordination of counting by two units (e.g., when students skip count 3 groups of 5, they will count by 5s while simultaneously keeping track of the number of groups as they hold up 1, 2, 3 fingers). **PO.SR.2b:** Difficulty: The hardest facts for students to "just know" are 6 x 8, 8 x 6, 7 x 8, 8 x 7, 8 x 8, 7 x 7, 7 x 6, 6 x 7.

[7] Use place value understanding and properties of operations to perform multi-digit arithmetic. Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

[8] 3.OA.9: Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.

Why Does the Math Make Sense? (PO.M)	How is This Concept Connected to Other Concepts? (PO.CC)	What Must I Understand About Teaching This Content? (PO.TC)	What Must I Understand About Student Reasoning in This Content? (PO.SR)

PO.M.1g: Explain why the commutative property of multiplication makes sense by subdividing rectangles and arrays in two different ways or by using geometric reasoning relating to area (Beckmann, 2011, p. 153).

PO.M.1h: Explain why the associative property of multiplication makes sense by deconstructing rectangular prisms built of unit cubes to show how to find the area of three distinct bases first while still multiplying the same three dimensions (Beckmann, 2011, p. 156).[2]

PO.M.1 i: Explain why the distributive property makes sense by describing the total number of objects in a subdivided array in two different ways and using expressions to describe both ways. Use simple situations to explain or illustrate the distributive property (Beckmann, 2011, p. 168).

PO.M.1j: Use the associative and commutative properties of multiplication and the distributive property of multiplication over addition to generate different solution methods for finding the product of two factors.

demonstrate how the dimensions of the different bases can be multiplied together first while still leading to the same volume.[9]

Connections to Future Learning

See future sub-categories in the Multiplication and Division: Building Procedural Fluency From Conceptual Understanding progression

PO.CC.2i: Provide examples to illustrate that the meaning of M*N as the number of units in M equal groups if there are N units in one group applies not only to whole numbers, but also to fractions/decimals (Beckmann, 2011, p. 220).

relationship between multiplication and division, the different ways the distributive property could be used to illustrate a single multiplication problem, etc.).

PO.TC.3d: Pose purposeful questions that invite students to make important mathematical connections among levels of reasoning or derived strategies based on student work.

PO.TC.3e: Recognize the language demands of a given task:

- Reading
- Writing
- Listening
- Speaking
- Representing
- Interacting

PO.TC.3f: Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development.

PO.SR.3: Student Supports

PO.SR.3a: Plan appropriate fluency supports for students with learning disabilities including research-based interventions such as:

- Identifying a small set of related arithmetic combinations[11]
- Doing quick, real-time review of prerequisite arithmetic combinations that might be helpful for the related facts
- Direct teaching of the new arithmetic combinations through multiple modalities (visual, symbolic, etc.).
- Guided practice with feedback
- Independent practice
- Assessment

(Agaliotis & Teli, 2016, p. 93)

[2] This competency is directly addressed in the grade 5 standards with the student work on volume. Grade 3 students will likely not use unit cubes to demonstrate associativity.

[9] Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths equivalently by multiplying the height by the area of the base. Represent three-fold whole-number products as volumes (e.g., to represent the associative property of multiplication).

[11] In the study cited, the term fact was replaced with “arithmetic combinations” to underscore the connections among facts — not just the facts in isolation.

Why Does the Math Make Sense? (PO.M)	How is This Concept Connected to Other Concepts? (PO.CC)	What Must I Understand About Teaching This Content? (PO.TC)	What Must I Understand About Student Reasoning in This Content? (PO.SR)
PO.M.1k: Derive fact strategies based on the foundational fact sets using visuals, symbols, and/or written explanations. Move to "just knowing" facts from memory (recall within three seconds): ● Concentrate on a small set of related, not yet mastered facts. ● Ensure that students can generate level 3 strategies for these facts before focusing on speed/accuracy. ● Engage in some kind of speed and accuracy practice (flashcards, a "sprint," a game, etc.) with real-time feedback. **PO.M.1l:** Build division fact strategies from multiplication fact strategies, including strategies such as: ● Dealing out (level 1) and move toward dealing out more efficient groups (level 2) in partitive division. ● Counting up in level 1 and keeping track of the number of groups with fingers (level 1) and skip counting up (level 2) in measurement division. (CCSS Writing Team, 2019, p.34) **PO.M.1m:** Demonstrate that an area of a rectangle, in square units, is the number of 1-unit-by-1-unit squares it takes to cover the rectangle		**PO.TC.3g**: Adapt common real-life contexts for multiplication, division, and properties of operations to connect to student prior knowledge and experience. **PO.TC.4 Recommended Tasks for Promoting Understanding** **PO.TC.4a:** Through contextualized or concrete application, identify multiplication (M x N) as the number of units (or objects) in M equal groups if there are N units (or objects) in one group[10] (Beckmann, 2011, p. 143). **PO.TC.4b:** Represent and solve contextualized multiplication and division problems that yield a range of developmentally representative strategies (such as drawing and counting every item drawn or drawing groups with numbers to represent a known collection while skip counting to find the total, etc.). **PO.TC.4c:** Represent arrays and use repeated addition or multiplication to show ways to find the total number of objects in the array. Recognize that arrays can be split into equal numbers of rows or	

[10] This ordering convention represents the common convention in the United States, though other countries have different orderings and interpretations.

Why Does the Math Make Sense? (PO.M)	How is This Concept Connected to Other Concepts? (PO.CC)	What Must I Understand About Teaching This Content? (PO.TC)	What Must I Understand About Student Reasoning in This Content? (PO.SR)
without gaps or overlaps. **PO.M.1n:** Explain that we can multiply L*W to find the area of an L-by-W rectangle because the rectangle can be subdivided into L rows, each of which contains W 1-by-1 unit squares (Beckmann, 2011, p. 159). **PO.M.1o:** Explain that area problems where regions are partitioned by unit squares are foundational for Grade 3 standards because "area is used to represent single-digit multiplication and division strategies, multi-digit multiplication and division in Grade 4, and multiplication and division of fractions in Grades 5 and 6" (CCSS Writing Team, 2019, p. 33). **PO.M.1p:** Become fluent with division facts (usually by leveraging knowledge of multiplication facts to think of division as finding a missing factor). **PO.M.2: Additional Math Content Knowledge for Teachers** **PO.M.2a:** Accurately represent situations involving equal groups multiplication, arrays, partitive division, and measurement division.		equal numbers of columns (Beckmann, 2011, p. 145). **PO.TC.4d:** Explain why the commutative property of multiplication makes sense by subdividing rectangles and arrays in two different ways or by using geometric reasoning relating to area (Beckmann, 2011, p. 153). **PO.TC.4e:** Mentally solve and reason about purposefully sequenced multiplication or division problems that leverage connections between known facts and derived strategies. **PO.TC.4f:** Explain why the distributive property makes sense by describing the total number of objects in a subdivided array in two different ways and using expressions to describe both ways. Use simple situations to explain or illustrate the distributive property (Beckmann, 2011, p. 168). **PO.TC.5 Common Instructional Misconceptions** **PO.TC.5a:** Identify common pitfalls in fluency instruction, including: • Misconception: Drill is the means by which students best learn their facts. Correction: Drill can lead to	

Why Does the Math Make Sense? (PO.M)	How is This Concept Connected to Other Concepts? (PO.CC)	What Must I Understand About Teaching This Content? (PO.TC)	What Must I Understand About Student Reasoning in This Content? (PO.SR)
PO.M.2b: Recognize whether a contextualized division problem is partitive or measurement division.[3] **PO.M.2c:** Identify the three levels of reasoning that students use in becoming fluent with the multiplication facts (e.g., doubling, adding a group to a known fact, etc.).[4] ● Level 1: Direct modeling of multiplication and division (e.g., 5 x 6 represented as 5 groups with 6 dots in each group) ● Level 2: Addition-based strategies (e.g., skip counting, using branching or grouping strategies, etc.) ● Level 3: Derived strategies based on the Associative Property and the Distributive Property (e.g., doubling and halving, adding a group to a known fact, etc.) (Carpenter et al., 1999, pp. 62- 63) **PO.M.2d:** Generate and sequence strategies for a given fact according to the levels of reasoning. Example 8 x 6: ● Level 1 examples: 8 groups of 6 dots ● Level 2 examples: 8 + 8 + 8 + 8 + 8 + 8 or 6, 12, 18, 24, 30, 36, 42, 48		incomplete or inaccurate memorization of information, lack of retention, and lack of transfer (Fennel, 2011, p. 32). ● Misconception: Students who lack fluency with their facts cannot proceed to more advanced concepts until they learn their facts. Correction: Students who don't know their multiplication facts can still engage in more challenging mathematical concepts. **PO.TC.5b:** Identify common pitfalls in properties of operations instruction, including: ● Misconception: The associative and commutative properties of multiplication and the distributive property are isolated rules and activities students must know how to perform to produce the correct answer. Correction: These tools are the means by which students work flexibly with numbers.	

[3] The heart of this module is on decontextualized operations with numbers; however, students develop different types of calculation strategies based on whether they are finding the number of groups or the number of units in each group. Teachers must be able to look at a contextualized problem and understand how the context of the problem will influence whether students use strategies like "dealing out" or skip-counting.

[4] These levels are the ones described in *Children's Mathematics: Cognitively Guided Instruction* and *The Progressions.* These levels follow a similar learning trajectory to Arthur Baroody's levels; however, they define student proficiency differently for levels 1, 2, and 3.

Why Does the Math Make Sense? (PO.M)	How is This Concept Connected to Other Concepts? (PO.CC)	What Must I Understand About Teaching This Content? (PO.TC)	What Must I Understand About Student Reasoning in This Content? (PO.SR)
• Level 3 examples: (5 x 8) + 8 = 40 + 8 = 48 or 8 x 6 = 4 x 12 = 48, 6 x 2 x 2 x 2 = 12 x 2 x 2 = 24 x 2 = 48 **PO.M.2e:** Using a series of expressions, demonstrate how two factors can be decomposed and regrouped using the associative property. **PO.M.2f:** Given a mental method of deriving a fact, write a series of mathematical expressions or equations that correspond to the method used and show which properties align with the reasoning (whether they were consciously applied or not) (Beckmann, 2011, p. 178). **PO.M.2g:** Use expressions or equations and decomposed arrays or area models to help describe how basic multiplication facts are related to other basic facts via properties of arithmetic (Beckmann, 2011, p. 178). **PO.M.2h:** Explain why we cannot divide a number by 0 (if m divided by n is 0, then 0 * n = m, which is impossible with the multiplication property of 0).		• Misconception: Order of operations is the agreed upon way to always approach simplifying expressions. Correction: Order of operations is a convention that provides consistency and clarity. However, given the expression 8 x (100 - 1), it may be more efficient to apply the distributive property than to simplify the expressions in parentheses. **PO.TC.5c:** Identify statements that are common but imprecise and explain why they are inaccurate using counterexamples: • Imprecise statement: “Multiplication is repeated addition.” Accurate statement: Repeated addition is one way of approaching finding the product of two whole numbers but does not encompass all rational number multiplication (McCallum, 2018, blog).	

What Must I Understand About My Curriculum's Approach to This Content? (PO.C)

PO.C.1 Understanding Curriculum Design

- **PO.C.1a:** Identify the vocabulary and definitions a curriculum uses for key concepts and procedures:
  - Multiplication
  - Division
    - Partitive division (e.g., equal sharing, etc.)
    - Measurement division (e.g., quotative division, etc.)
  - Array
  - Area
  - Fact fluency strategies (e.g., doubling, doubling & halving, adding a group, etc.)
- **PO.C.1b:** Identify real-life contexts used consistently with key concepts and procedures:
  - Multiplication (e.g., wheels on a car, eggs in a basket, etc.)
  - Division (e.g., wheels on a car, eggs in a basket, etc.)
- **PO.C.1c:** Identify which visual representations a curriculum uses to represent key concepts and procedures:
  - Multiplication (e.g., number lines, rekenreks, tape diagrams, etc.)
  - Division (e.g., number lines, rekenreks, tape diagrams, etc.)
- **PO.C.1d:** Identify the language and symbolic notation a curriculum uses to emphasize the following key understandings:
  - Multiplication (e.g., x, *, "next to notation," etc.)
  - Division (e.g., $\div$, $\overline{)}$, etc.)
  - The Associative Property (e.g., Are grouping symbols used? Is unit form used?)

PO.C.2 Task Analysis

PO.C.2a: Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

Grade 3:

- We can represent and solve equal groups contextualized multiplication and division problems using a range of strategies from drawing every group and the number of units in every group to composing more efficient visual representations like area models to writing expressions or equations to explain reasoning.
- We can split arrays into equal numbers of rows or equal numbers of columns.
- Skip counting is a natural strategy to solve a measurement division problem; dealing out is a natural strategy to solve a partitive division problem.
- The commutative property of multiplication makes sense because we can subdivide rectangles and arrays in two different ways.
- The distributive property makes sense because it describes the total number of objects in a subdivided array in different ways and can be explained using different expressions.

- We can derive unknown math facts through applying the arithmetic properties to known facts.
- We can use a series of mathematical expressions or equations to demonstrate how the properties of operations are used to derive strategies.

Grades 4/5:

- The associative and distributive properties enable us to efficiently multiply large numbers by decomposing factors into base ten units and multiplying the base ten units to find partial products.
- The distributive property enables us to perform mental calculations beyond the facts known from memory.

Grade 5:

- We can demonstrate the associative property by building rectangular prisms from unit cubes and using grouping symbols to demonstrate how the dimensions of the different bases can be multiplied together first while still leading to the same volume.

PO.C.3 Coherence

PO.C.3a: Determine where and how a given curriculum addresses the following standards and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematical connection to the properties of operations explicit.

Grade 3:

- 3.OA.9: Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into 2 equal addends.
- 3.NBT.3: Use place value understanding and properties of operations to perform multi-digit arithmetic. Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations. (A range of algorithms may be used.)
- 3.MD.7: Geometric measurement: understand concepts of area and relate area to multiplication and to addition. Relate area to the operations of multiplication and addition.
- 3.MD.7a: Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
- 3.MD.7c: Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

Grade 5:

- 5.MD.5a: Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths equivalently by multiplying the height by the area of the base. Represent three-fold whole-number products as volumes, e.g., to represent the associative property of multiplication.

Conceptual Foundations of Multi-Digit Multiplication (MDM)

This content competency extends the work of single-digit multiplication to generalized multi-digit multiplication. Research indicates significant time should be spent on the conceptual foundations rooted in visual representations and decompositions of factors into base ten units. From there, students should apply understanding of place value and properties of operations to generalize more efficient strategies and eventually algorithms (CCSS Writing Team, 2019) (National Research Council, 2001). In order to become proficient with algorithms, "Students need opportunities to practice on a moderate number of carefully selected problems after they have established a strong conceptual foundation and the ability to explain the mathematical basis for a strategy or procedure. At that point, providing students with practice on a small number of problems, "spacing" or distributing these over time, and including feedback on student performance support learning outcomes" (National Council of Teachers of Mathematics, 2014, p. 45).

Focus Standards

3.NBT.A.3: Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

4.NBT.A.1: Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.

4.NBT.B.5: Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

5.NBT.A.1: Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.

5.NBT.B.5: Fluently multiply multi-digit whole numbers using the standard algorithm.

5.NBT.B.7: Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category

4.NBT.B.6: Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

5.NBT.B.6: Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

5.NF.B.4.b: Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Why Does the Math Make Sense? (MDM.M)	How is This Concept Connected to Other Concepts? (MDM.CC)	What Must I Understand About Teaching This Content? (MDM.TC)	What Must I Understand About Student Reasoning in This Content? (MDM.SR)

MDM.M.1: Understanding the Concept Progression

Grade 3:

MDM.M.1a: Write a series of expressions to demonstrate how the associative property enables us to multiply multiples of 10 (e.g., 4 x 60 = 4 x (6 x 10) = (4 x 6) x 10 = 24 x 10 = 240).

Grade 4:

MDM.M.1b: Use place value reasoning, math drawings, and/or arrays/area models to explain patterns in the products where one factor increases by powers of 10 such as: 6 x 7, 6 x 70, 6 x 700, and 6 x 7000 (CCSS Writing Team, 2019, p.65).

MDM.M.1c: Describe multiplication by 10 as moving digits one place to the left and use math drawings of bundled objects to support the explanation (Beckmann, 2011, p. 151).

MDM.M.1d: Explain how many places each digit will shift based on which power of ten it is multiplying a given number.

MDM.M.1e: Construct an area model out of place value blocks or grid paper showing the decomposition of each factor into base-ten units and the resulting partial products.

MDM.M.1f: Decompose whole numbers into expanded form and draw and explain array/area

MDM.CC.1: Connections to Prior Learning

See prior sub-categories in the Multiplication and Division: Building Procedural Fluency From Conceptual Understanding progression

MDM.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category

Grade 4:

MDM.CC.2a: Explain how using the division method of using area models to find a missing side length by subtracting partial quotients connects to finding a product of two factors by using area models to find and add partial products.

Grade 5:

MDM.CC.2b: Find the area of a rectangle with fractional or decimal side lengths by creating area models and finding the area of the partial products before summing them to find the area using area models such as the one below.

MDM.TC.1: Visual Diagrams and Representations

MDM.TC.1a: Understand the research supports sustained practice with a consistent concrete model that illustrates the process of bundling/unbundling tens as long as the concrete model is directly linked to the steps carried out in symbolic form (National Research Council, 2001, p. 198).

MDM.TC.1b: Compare the benefits of the different representations that demonstrate the uniformity in our base ten system, such as:

- The base ten diagrams (either strips as shown below or diagrams that resemble base ten blocks) are proportional to the size of the place represented, show the relative magnitude of each digit, and show the relationship within each base ten unit (so we can simultaneously see that 1 hundred block is the same as 10 tens is the same as 100 ones, which is useful for reinforcing bundling/unbundling) (National Research Council, 2001, p. 198).

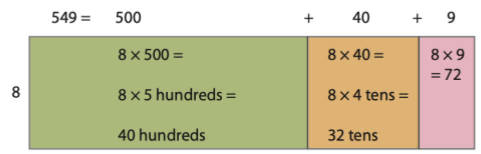
MDM.SR.1: Student Misconceptions and Difficulties

MDM.SR.1a: Recognize the challenge students face when constructing area models using base ten blocks, including:

- Difficulty: Length and width of the rectangle are not represented by the actual blocks but their dimensions.
- Difficulty: Though students usually begin with repeated rows or columns (like constructing 12 rows of 2 tens and 3 ones below), they eventually generalize placing hundreds blocks to show ten unit by ten unit partial products; this is not apparent to students beginning with base ten blocks.
- Difficulty: Forming two-digit by two-digit factor arrays with the most efficient use of base ten blocks requires turning some of the base ten blocks in 90 degree rotations.

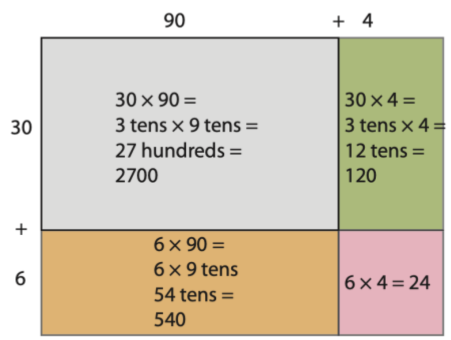
Why Does the Math Make Sense? (MDM.M)

models while connecting the partial products in the area model to the partial products written strategy.



$$8 \times 549 = 8 \times (500 + 40 + 9) = 8 \times 500 + 8 \times 40 + 8 \times 9$$

(Fuson & Beckmann, 2012, p. 23)

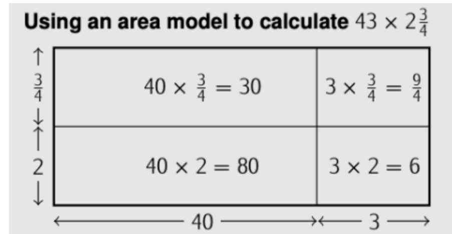


$$36 \times 94 = (30 + 6)\times(90 + 4) = 30\times90 + 30\times4 + 6\times90 + 6\times4$$

(Fuson & Beckmann, 2012, p. 24)

MDM.M.1g: Given two- or three-digit whole number factors, write equations that use expanded forms and the distributive property to solve. Relate the equations to the partial products in an array/area model or in written partial products (Beckmann, 2011, p. 192).

How is This Concept Connected to Other Concepts? (MDM.CC)



(CCSS Writing Team, 2019, p.153)

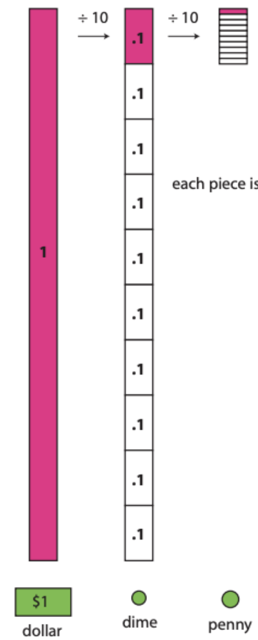
MDM.CC.3: Connections to Future Learning

MDM.CC.3a: Explain how the distributive property is applied in algebraic thinking, including:

- Generating equivalent algebraic expressions using the distributive property.

Explaining why procedures like "FOIL" work to multiply two binomial expressions using area models and expressions. Connect the area model and expressions to the distributive property.

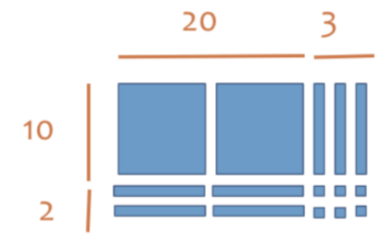
What Must I Understand About Teaching This Content? (MDM.TC)



(CCSS Writing Team, 2019, p.153)

- The place value disks clearly demonstrate the process of bundling and unbundling and are also generalizable across many places.

What Must I Understand About Student Reasoning in This Content? (MDM.SR)



MDM.SR.1b: Misconception: Students forget to multiply all partial products (e.g., 32 x 33 = 906).

MDM.SR.1c: Misconception: Student has set up multi-digit multiplication and forgotten to multiply each set of partial products based on place value and instead multiplied the digits without considering place value.

$$\begin{array}{r} 549 \\ \times\ 8 \\ \hline 72 \\ 32 \\ 40 \\ \hline 144 \end{array}$$

MDM.SR.1d: Misconception: Student has added on a 0 to a decimal number to indicate it has been multiplied by 10 (e.g., 3.4 x 10 = 3.40).

MDM.SR.1e: Misconception: Students may confuse the direction that a decimal "moves" and the direction that digits "move" on a place value chart when

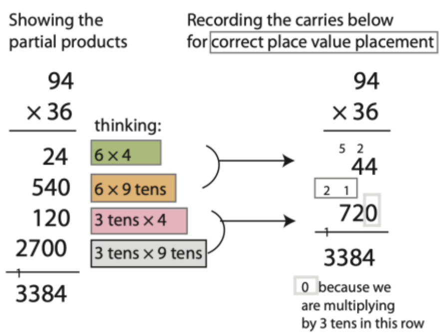
Why Does the Math Make Sense? (MDM.M)	How is This Concept Connected to Other Concepts? (MDM.CC)	What Must I Understand About Teaching This Content? (MDM.TC)	What Must I Understand About Student Reasoning in This Content? (MDM.SR)

Grade 5:

MDM.M.1h: Decompose decimal numbers into expanded form and draw and explain array/area models while connecting the partial products in the area model to the partial products written strategy.

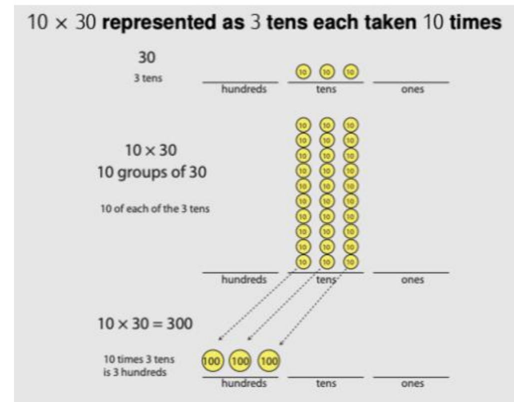
MDM.M.1i: Understand that the standard multiplication algorithm can be explained by the definition of multiplication, the structure of our base-ten system, and the properties of arithmetic (Beckmann, 2011, p. 195).

MDM.M.1j: Explain how recording carries allows students to write partial products more efficiently.



(Fuson & Beckmann, 2012, p. 24)

MDM.M.1k: Justify each step of the standard algorithm (using a three-digit factor and a two-digit factor) using the language of place value and properties of operations.



(CCSS Writing Team, 2019, p.153)

MDM.TC.1c: Compare the benefits of area models subdivided into base ten unit partial products and area models subdivided into partial products based on the lengths written in expanded form, such as:

- The area model subdivided into base ten units clearly shows each partial product can be represented using base ten blocks and translates well to unit form (e.g., 6 tens x 4 tens is 24 tens).
- The area model on the right can be drawn efficiently and the partial products are more clearly connected to the symbolic form of partial products and the standard algorithm.

multiplied by a power of 10 (e.g., 3.4 x 10 would result in the decimal point moving one place to the right or in the digits moving one place to the left on a place value chart).

MDM.SR.1f: Difficulty: Recognize that students do not intuit the decimal point is often dropped from whole numbers but would otherwise be located after the ones place.

MDM.SR.1g: Difficulty: Recognize that students may not understand that shifts to the left in place value that result from multiplying a number by powers of 10 may require the student to use 0s to occupy the places immediately to the left of the decimal point.

MDM.SR.2: Alternative Methods of Reasoning

MDM.SR.2a: Explain why the lattice method of multiplication works using explanations grounded in place value. (The value of the digits in each grid square is determined by the place value of the contributing factors and the diagonals automatically regroup each product into the next highest place value.)

Why Does the Math Make Sense? (MDM.M)	How is This Concept Connected to Other Concepts? (MDM.CC)	What Must I Understand About Teaching This Content? (MDM.TC)	What Must I Understand About Student Reasoning in This Content? (MDM.SR)

MDM.M.1l: Multiply decimal numbers by treating them as whole numbers and using estimation to accurately place the decimal point in the product (CCSS Writing Team, 2019, p.69).

MDM.M.1m: Multiply decimal numbers by treating them as whole numbers and explain where to place the decimal point in the product using an explanation grounded in place value (CCSS Writing Team, 2019, p.69).

MDM.M.1n: Multiply decimal numbers by recording the factors as mixed numbers with decimal fractions and using area models to find the partial products.

MDM.M.2: Additional Math Content Knowledge for Teachers

MDM.M.2a: Justify the steps of the standard algorithm in multiplying a three-digit factor by two-digit factor whole number multiplication.

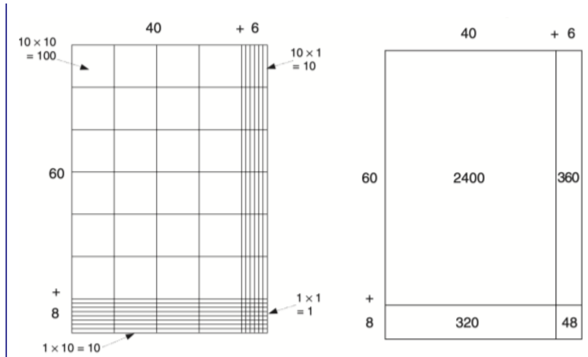
MDM.M.2b: Using a series of expressions, demonstrate how the properties of operations enable us to multiply a two-digit by two-digit number by primarily leveraging our single digit math facts.

For example:

36 x 94 =

(30 + 6) x 94 =

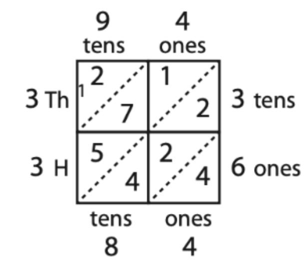
39 x 94 + 6 x 94 =



(National Research Council, 2001, p. 211)

MDM.TC.1d: Compare the benefits of the different methods for drawing area models, such as:

- The area model on the left shows the factors decomposed into tens and then ones and more closely resembles the structure of area models students work with through algebra.
- The area model on the right more naturally matches the steps of recording in the standard algorithm (e.g., to multiply 94 x 36, we would multiply 94 through by 6 before recording our second line of partial products as 94 x 30).



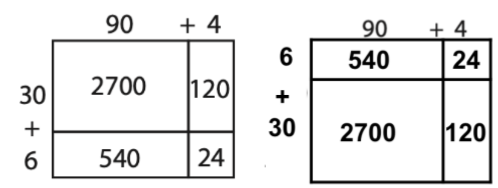
(Fuson & Beckmann, 2012, p. 24)

MDM.SR.3: Student Supports

MDM.SR.3a: Plan appropriate scaffolds for multi-digit multiplication for students with learning disabilities including research-based practices such as:

- Providing a conceptual preview using manipulatives or diagrams (such as area models using grid paper)
- Recording and connecting every step of the symbolic procedure to the conceptual preview of the same problem
- Practicing one or two problems each day where students provide conceptual explanations to support their work

(Fennell, 2011, p. 115)

Why Does the Math Make Sense? (MDM.M)	How is This Concept Connected to Other Concepts? (MDM.CC)	What Must I Understand About Teaching This Content? (MDM.TC)	What Must I Understand About Student Reasoning in This Content? (MDM.SR)
30 x (90 + 4) + 6 x (90 + 4)= 30 x 90 + 30 x 4 + 6 x 90 + 6 x 4 = **MDM.M.2c:** Justify the steps of the standard algorithm in multiplying a three-digit factor by two-digit factor whole number multiplication.		 (image on left from Fuson & Beckmann, 2012, p. 24) **MDM.TC.2: Symbolic Diagrams and Representations** **MDM.TC.2a:** Compare the benefits of the two methods for recording partial products, such as: • The method on the left works from left to right (which is how we read) and provides an immediate estimation of how large the product will be. • The method on the right more closely resembles the order of multiplication in the standard algorithm. • Both methods allow students to record each partial product separately without regrouping.	**MDM.SR.3b**: Understand the difficulty of alternating multiplication and addition in our standard multiplication algorithms; leverage area models as a method that involves multiplying first and then adding partial products later in two distinct phases (National Research Council, 2001, p. 208). **MDM.SR.3c**: Leverage grid paper with darkened ten by ten square outlines as a resource for students to represent area models while seeing the role of the base ten units in the partial products. This resource also enables students to see the individual unit squares that make up the total area while allowing them to count or find the total in more efficient ways, such as adding individual base ten units. **MDM.SR.3d**: Provide the "helping step" of writing decomposed factors in the standard algorithm next to the factors (CCSS Writing Team, 2019, p.65).

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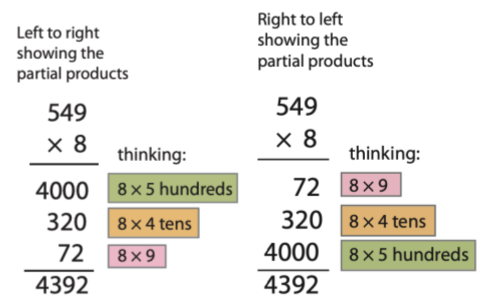

(Fuson & Beckmann, 2012, p. 23)

MDM.TC.2b: Compare the benefits of the two methods for recording carries, such as:

The method on the left organizes all carries into their place value whereas the method on the right more closely resembles the carries in the traditional algorithm.
The method on the right also has fewer numbers to then organize for addition of the partial products (which would be necessary for multi-digit by multi-digit multiplication).

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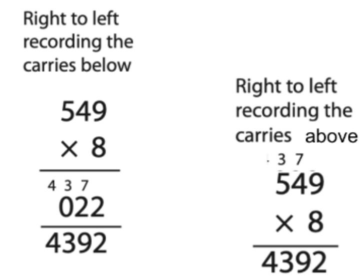
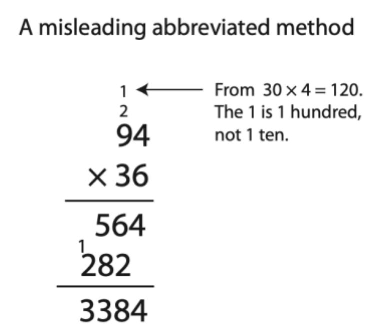
Why Does the Math Make Sense? (MDM.M)	How is This Concept Connected to Other Concepts? (MDM.CC)	What Must I Understand About Teaching This Content? (MDM.TC)	What Must I Understand About Student Reasoning in This Content? (MDM.SR)



(Fuson & Beckmann, 2012, p. 23)

MDM.TC.2c: Explain why the traditional approach of recording newly composed units above the top factor sometimes violates the convention that a number recorded in a given place should represent that place (CCSS Writing Team, 2019, p.65).



(Fuson & Beckmann, 2012, p. 25)

MDM.TC.2d: Explain the benefits of rewriting decimal numbers as mixed numbers, including:

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		• Decimal fractions help students connect to knowledge of money (3/10 can be thought of as 3 dimes or equivalently as 30/100 or 30 cents) to underscore equivalency (CCSS Writing Team, 2019, p.148). • Students can more easily see the product of $\frac{1}{10} \times \frac{1}{10}$ than 0.1 x 0.1 when finding partial products in area models. **MDM.TC.3: Recommended Instructional Competencies** **MDM.TC.3a**: Select and sequence student work that could highlight an important mathematical connection (such as the relationship between multiplication and division, the relationship between the area model and the procedural steps, etc.). **MDM.TC.3b**: Pose purposeful questions that invite students to make important mathematical connections among different representations of multi-digit multiplication. **MDM.TC.3c**: Make the mathematics of the lesson explicit through the use of explanations, representations, tasks, and/or examples.	

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		MDM.TC.3d: Address a common misconception by engaging students in analysis of a task that illustrates the misconception. **MDM.TC.3e**: Recognize the language demands of a given task: ● Reading ● Writing ● Listening ● Speaking ● Representing ● Interacting **MDM.TC.3f**: Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development. **MDM.TC.3g**: Adapt common real-life contexts for multiplication to connect to student prior knowledge and experience. **MDM.TC.4: Recommended Tasks for Promoting Understanding** **MDM.TC.4a**: Use objects or pictures to explain how we use the associative property to mentally multiply multiples of 10 (such as 4 x 60). **MDM.TC.4b**: Explain whether two different	

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		expressions resembling the distributive property are equivalent by drawing arrays. For example, 8 x (5 + 1) = 8 x 5 + 8 x 1. **MDM.TC.4c**: Subdivide an array and write two expressions that could be used to find the total (e.g., 17 x 5 could be subdivided into 10 + 7 and 5 and recorded as (10 + 7) x 5 and 10 x 5 + 7 x 5). **MDM.TC.4d**: Mentally solve and reason about purposefully-sequenced multiplication problems that leverage the properties of operations (e.g., 5 x 19, 5 x 190, etc.). **MDM.TC.4e**: Given an array with between 20 and 80 rows and columns, use various strategies to find the total. (Students will likely start with counting all or repeated addition before finding 10s and eventually finding larger partial products.) **MDM.TC.4f**: Given a two-digit by two-digit whole number multiplication problem, create an area model using grid paper (with or without darkened lines around squares of one hundred). **MDM.TC.4g**: Connect the partial products recorded in an area model to the partial products in written methods. **MDM.TC.4h**: Given a product that is a multiple of 10, generate different possible	

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		factors (e.g., how many pairs of factors can you think of that multiply to 24,000?). **MDM.TC.4i**: Given a product that is a two-digit decimal number, generate different possible factors (e.g., how many pairs of factors can you think of that multiply to 2.4?). **MDM.TC.4j**: Explain how the partial products in an area model relate to the partial products in written methods of multiplication (both partial products and standard algorithm). **MDM.TC.4k**: Upon readiness to practice the algorithm, practice a limited set with feedback. **MDM.TC.5: Common Instructional Misconceptions** **MDM.TC.5a**: Identify statements that are common but imprecise and explain why they are inaccurate: • Imprecise statement: "To multiply a number by 10, put a 0 at the end of the number." Rationale: This does not hold true for decimals or non-base ten numbers such as fractions (Beckmann, 2011, p. 151). • Imprecise statement: "The number of 0s at the end of a product is	

Why Does the Math Make Sense? (MDM.M)	How is This Concept Connected to Other Concepts? (MDM.CC)	What Must I Understand About Teaching This Content? (MDM.TC)	What Must I Understand About Student Reasoning in This Content? (MDM.SR)
		determined by the number of 0s at the end of each factor." Rationale: This does not hold true for multiplying 40 and 50 since 40 x 50 does not equal 200 (CCSS Writing Team, 2019, p.65). • Imprecise statement: "Multiplying 10s by moving to the right." Rationale: This language is ambiguous in that the placement of a decimal will shift one place to the right every time a number is multiplied by 10, but the digits themselves will move to the next greatest place value which could be illustrated by moving to the left on a place value chart. • Imprecise statement: "When we multiply two decimal numbers, the decimal point in the product will always be placed in as many spaces from the right as the sum of the position of the decimal point from the right in the two factors." Rationale: This is misleading because the product of 1.4 and 2.5 is more often written as 4.9 than 4.90. • Imprecise statement: "Multiplying numbers together always produces a bigger number." Rationale: This is	

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		not true when we multiply by values less than or equal to 1. **MDM.TC.5b**: Identify common pitfalls in multi-digit multiplication instruction, including: • Misconception: Students should transition to the algorithm before students have solid conceptual understanding of expanded algorithms. Correction: Students should spend ample time building a conceptual foundation before transitioning to the algorithm.	

MDM.C.1: Understanding Curriculum Design

- **MDM.C.1a:** Identify the vocabulary and definitions a curriculum uses for key concepts and procedures:
    - The Distributive Property
    - The Associative Property
    - Creating a new unit from ten of an existing unit (e.g., bundling, regrouping, etc.)
- **MDM.C.1b**: Identify real-life contexts used consistently with key concepts and procedures:
    - Multiplication (e.g., rows in a theatre, buses of students, etc.)
- **MDM.C.1c:** Identify which visual representations a curriculum uses to represent key concepts and procedures:
    - Multiplication (e.g., arrays, area models, place value disks, place value blocks, etc.)
- **MDM.C.1d**: Identify the language, structures, and symbolic notation a curriculum uses to emphasize the following key understandings:
    - Multiplication
        - Area Models
            - How is the process of using an area model to multiply two factors explained?
            - Are area models constructed with base ten units initially?
            - How are they structured? (E.g., greatest place farthest left across on top on vertical dimension? Or greatest place on bottom on vertical dimension?)
            - Where is the multiplication leading to partial products recorded? Is it recorded in numeric form or unit form?
            - Are dimensions recorded in expanded form numerically or in unit form?
            - Where are the partial products recorded?
        - Written Partial Products
            - How is the process of using written partial products to multiply two factors explained?
            - Are the factors rewritten in expanded form or left in standard form?
            - In what order is the multiplication performed? (E.g., largest place first or last?)
            - Is there an organizing structure where students write the multiplication on the side? Are numbers written in unit form or numerically?
        - Algorithms
            - How is the process of using an area model to multiply two factors explained?
            - How and where are "carries" recorded?
            - How are the zeros at the end of the second and greater lines of partial products recorded? (E.g., some curricula leave blank spaces.)
            - In decimal multiplication, how is the initial adjustment to whole number factors described? (E.g., are arrows shown to indicate how many powers of 10 each factor was multiplied by to produce a whole number?)
            - In decimal multiplication, how is the final adjustment to place the decimal in the product described? (E.g., are arrows shown to indicate how many powers of 10 the product should be divided by to accurately place the decimal?)

MDM.C.2: Task Analysis

MDM.C.2a: Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

Grade 3:

- The associative property enables us to efficiently multiply multiples of 10.

Grade 4:

- Our place value system explains patterns in the products where one factor increases by powers of 10 such as: 6 x 7, 6x 70, 6 x 700, and 6 x 7000.
- We can describe multiplication by 10 as moving digits one place to the left.
- We can decompose area models with the same dimensions in different ways and the partial products should still total the same product.
- We can visually show how we use the distributive property to multiply multi-digit numbers by decomposing arrays or area models and summing the partial products.
- We can relate calculations used to find the partial products shown in an area model to the calculations used to find the partial products shown in written methods.

Grade 5:

- We multiply decimal numbers by treating them as whole numbers and can use estimation to accurately place the decimal point in the product.

MDM.C.2b: Select tasks from a curriculum that will surface the following misconceptions and craft the mathematical idea that will help students make sense of a misconception:

Grade 4:

- Recognize when a student has forgotten to multiply all partial products (e.g., 32 x 33 = 906).

Grade 5:

- Recognize when a student has forgotten to multiply all partial products (e.g., 32 x 33 = 906).
- Recognize that students may not understand that shifts to the left in place value that result from multiplying a number by powers of 10 may require the student to use 0s to occupy the places immediately to the left of the decimal point.
- Recognize when a student has added on a 0 to a decimal number to indicate it has been multiplied by 10 (e.g., 3.4 x 10 = 3.40).

MDM.C.3: Coherence

MDM.C.3a: Determine where and how a given curriculum addresses the following standards in grades 4-5 and plan coherent connections in area models used in these standards to the role of area models in base ten multiplication:

- 4.NBT.B.6: Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
- 5.NBT.B.6: Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
- 5.NF.B.4.b: Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Conceptual Foundations of Multi-Digit Division (MDD)

Division is one of the most challenging concepts for students in grades 3 - 5, so much so that grade 5 performance on division is one of the leading indicators of future success in high school (Siegler et al., 2012). Division presents unique challenges in that it answers different conceptual questions — how many units in one group? how many groups? given a known area of a rectangle and a side length, what is the missing side length? — which then inform the language we use to justify our algorithms — how many tens can we place in each of 7 groups? how many 10s of 7 can be made from a given number? how many side lengths of 10 units can be made from the remaining area? Moreover, the algorithm requires a degree of fluency in all four operations as well as exact calculations to determine each digit by place value in the quotient. The precision of language and conceptual understanding required to teach division effectively are laid out in the progression below. As with multiplication, taking significant time to develop conceptual understanding through connecting visual models with expanded algorithms is essential to developing long term procedural fluency.

Focus Standards
4.NBT.A.1: Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.
4.NBT.B.6: Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
5.NBT.A.1: Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
5.NBT.B.6: Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
5.NBT.B.7: Add, subtract, multiply, and divide decimals to hundredths using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category
4.NBT.B.5: Multiply a whole number of up to four digits by a one-digit whole number, and multiply two-digit numbers using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
5.NBT.B.5: Fluently multiply multi-digit whole numbers using the standard algorithm.
5.NF.B.7: Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. **5.NF.B.7a:** Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3. **5.NF.B.7b:** Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4.

Why Does the Math Make Sense? (MDD.M)	How is This Concept Connected to Other Concepts? (MDD.CC)	What Must I Understand About Teaching This Content? (MDD.TC)	What Must I Understand About Student Reasoning in This Content? (MDD.SR)

Why Does the Math Make Sense? (MDD.M)

Grades 4-5[1]:

MDD.M.1: Understanding the Concept Progression

MDD.M.1a: Analyze and explain the role of place value and the associative property in dividing numbers by 10 and then multiples of 10.

MDD.M.1b: Use base ten blocks to enact multi-digit by single digit division as finding the number of units in each group[2] and relating each step to the partial quotients symbolic method.

MDD.M.1c: Demonstrate the "finding the number of units in each group" approach by using representations of base ten units like place value charts and record and describe each step using the partial quotients symbolic method.

MDD.M.1d: Demonstrate the "finding the number of units in each group" partial quotients methods by subtracting out groups multiplied by base ten units.

MDD.M.1e: Demonstrate the partial quotients approach by multiplying the divisor by more efficiently grouped units. (The example on the left shows units grouped more efficiently than 10s and 1s but not as efficiently as the method on the right.)

How is This Concept Connected to Other Concepts? (MDD.CC)

MDD.CC.1: Connections to Prior Learning

See prior sub-categories in the Multiplication and Division: Building Procedural Fluency From Conceptual Understanding progression

MDD.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to this Module

Grade 5:

MDD.CC.2a: Rewrite decimal base ten units as decimal fractions and compute quotients of whole numbers and decimal fractions (and vice versa).

MDD.CC.2b: Given an area model with some missing side length segments and partial products, fill in the missing values.

What Must I Understand About Teaching This Content? (MDD.TC)

MDD.TC.1: Symbolic Diagrams and Representations

MDD.TC.1a: Compare the benefits of recording partial quotients using the methods below, such as:

- The method on the left records partial quotients in a way that will help students make sense of what the digits mean in the quotient using the standard algorithm.
- The method on the right helps students connect the partial quotients to each amount being subtracted and also takes up less space.

$$\begin{array}{r} 8 \\ 30 \\ 100 \end{array}\Big\} 138 \qquad \begin{array}{r} 7\overline{)966} \\ -700 \\ \hline 266 \\ -210 \\ \hline 56 \\ -56 \\ \hline 0 \end{array}$$

$$\begin{array}{r|l} 46\overline{)3129} & \\ -2300 & 50 \\ \hline 829 & \\ -460 & 10 \\ \hline 369 & \\ -230 & 5 \\ \hline 139 & \\ -92 & 2 \\ \hline 47 & \\ -46 & 1 \\ \hline \text{R }1 & 68 \end{array}$$

What Must I Understand About Student Reasoning in This Content? (MDD.SR)

MDD.SR.1: Student Misconceptions and Difficulties

MDD.SR.1a: Identify challenges students face with our traditional long division algorithm, such as:

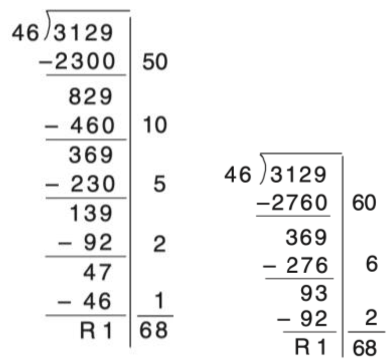
- Difficulty: The algorithm requires students to determine exactly the maximum copies of the divisor that they can take from the dividend.
- Difficulty: The algorithm creates no sense of the size of the answers that students are writing (Fuson, 2003, pp. 303 - 304).
- Difficulty: The algorithm requires fluency in all four operations.

MDD.SR.1b: Difficulty: Zeros in the dividend, in the remainder part way through division, or in the quotient all are common challenges students face with division.

[1] Both grades 4 and 5 division standards call for strategies based on place value, properties of operations, and the relationship between multiplication and division. Grade 4 division is restricted to single digit divisors, whereas grade 5 standards allow for two-digit divisors.

[2] Division as finding the number of groups is not covered in this section as our algorithms typically build from division as finding the number of units in each group. That said, teachers are expected to be able to justify both in the "Additional Teacher Content Knowledge" section.

Why Does the Math Make Sense? (MDD.M)



(National Research Council, 2001, p. 211)

MDD.M.1f: Use base ten blocks to enact multi-digit by single-digit division as finding a missing side length and record and describe each step using the partial quotients symbolic method.

MDD.M.1g: Use grid paper to demonstrate multi-digit by single-digit division as finding a missing side length and record and describe each step using the partial quotients symbolic method.

MDD.M.1h: Use an area model to demonstrate multi-digit by single-digit division as finding a missing side length by using base ten unit side lengths and record and describe each step using the partial quotients symbolic method (e.g., 496 ÷ 8 would be side lengths of 10 + 10 + 10 +10 + 10 + 10 + 1 + 1).

How is This Concept Connected to Other Concepts? (MDD.CC)

MDD.CC.3: Connections to Future Learning

MDD.CC.3a: Use long division to write rational numbers as decimals. Recognize that the decimal form of a rational number terminates in 0s or eventually repeats.[3]

What Must I Understand About Teaching This Content? (MDD.TC)

(CCSS Writing Team, 2019, p.153; National Research Council, 2001, p. 211)

MDD.TC.2: Recommended Instructional Competencies

MDD.TC.2a: Select and sequence student work that could highlight an important mathematical connection (such as transitioning from subtracting out groups multiplied by base ten units to subtracting out groups multiplied by more efficiently grouped units in partial quotient division).

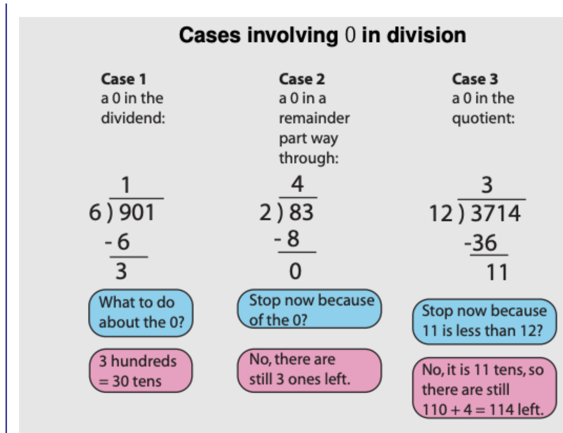
MDD.TC.2b: Pose purposeful questions that invite students to make important mathematical connections among different representations of multi-digit division.

MDD.TC.2b: Elicit mathematical explanations about division linking division concepts to procedures.

MDD.TC.2c: Make the mathematics of the lesson explicit through the use of explanations, representations, tasks, and/or examples.

MDD.TC.2d: Address a common misconception by engaging students in

What Must I Understand About Student Reasoning in This Content? (MDD.SR)



MDD.SR.2: Student Supports

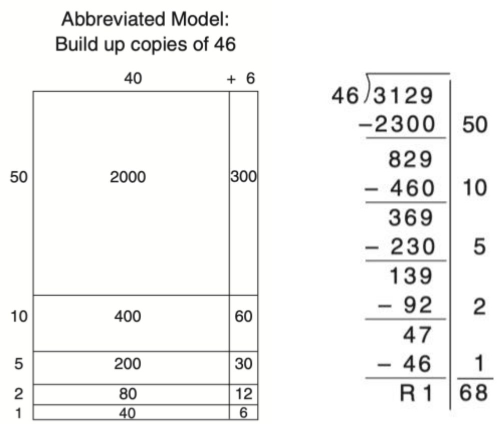
MDD.SR.2a: Plan appropriate scaffolds for multi-digit division for students with learning disabilities including research-based practices such as:

- Providing a conceptual preview using manipulatives or diagrams (such as area models using grid paper)
- Recording and connecting every step of the symbolic procedure to the conceptual preview of the same problem

[3] 7.NS.A.2d: Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Why Does the Math Make Sense? (MDD.M)	How is This Concept Connected to Other Concepts? (MDD.CC)	What Must I Understand About Teaching This Content? (MDD.TC)	What Must I Understand About Student Reasoning in This Content? (MDD.SR)

MDD.M.1i: Use an area model to enact multi-digit by single-digit division as finding a missing side length by using more efficiently grouped side lengths and record and describe each step using the partial quotients symbolic method (e.g., 2,720 ÷ 46 would be side lengths of 50 + 10 + 5 + 2 + 1).



(National Research Council, 2001, p. 211)

MDD.M.1j: Apply the "underestimate" strategy by rounding up a divisor to the nearest place. Explain why this strategy is effective in partial quotients division (Fuson & Beckmann, 2012, p. 27).

Grade 6:

MDD.M.1k: Justify the steps to the standard division algorithm for division using properties.

analysis of a task that illustrates the misconception.

MDD.TC.2e: Recognize the language demands of a given task:

- Reading
- Writing
- Listening
- Speaking
- Representing
- Interacting

MDD.TC.2f: Plan to enact tasks addressing multiple language modalities. Apply the Mathematical Language Routines to existing tasks while promoting language access and development.

MDD.TC.2g: Adapt common real-life contexts for division to connect to student prior knowledge and experience.

MDD.TC.3: Recommended Tasks for Promoting Understanding

MDD.TC.3a: Use objects or pictures to explain why we shift each digit's place value down when we divide by 10.

MDD.TC.3b: Mentally solve and reason about purposefully-sequenced division problems that leverage the properties of operations (e.g., 100 divided by 5, 200 divided by 5, 50

- Practicing one or two problems each day where students provide conceptual explanations to support their work

  (Fennell, 2011, p. 115)

MDD.SR.2b: Provide structured questions and sentence stems that guide students through the division language used by their curriculum (e.g., "How many units can we place in each group?" and "There are __ hundreds in each group. There are ____ tens in each group. There are ____ ones in each group.").

MDD.SR.2c: Recognize that partial quotient approaches allow students to produce underestimates which is helpful to students who struggle to produce the maximum partial quotient in each step of the division algorithm.

MDD.SR.2d: Provide grid paper with or without darkened lines around hundreds squares to allow students to represent division as finding a missing side length.

Why Does the Math Make Sense? (MDD.M)	How is This Concept Connected to Other Concepts? (MDD.CC)	What Must I Understand About Teaching This Content? (MDD.TC)	What Must I Understand About Student Reasoning in This Content? (MDD.SR)
MDD.M.2: Additional Math Content Knowledge for Teachers **M.DD.M.2a**: Explain the two interpretations of division by describing a context for each intepreration and drawing an accurate visual representation: ● How many groups? ● How many units in one group? **M.DD.M.2a**: For both types of division, justify the steps to partial quotient division using language that reinforces the type of division, such as: ● 496 ÷ 8in partitive division language answers, "How many units can we place in each of 8 groups?" Lines of reasoning would include, "We can place 6 tens in each group." ● 496 ÷ 8in measurement division language answers, "How many 8s are in 496s?" Lines of reasoning would include, "There are 6 tens of 8 in each group." (Beckmann, 2011, pp. 240, 242) **MDD.M.2b**: Recognize that most curricula adopt either partitive or measurement division language and reasoning while also teaching division as a missing side length. Given a task from a curriculum, identify whether their approach answers "how many groups?" measurement division or "how many units in one group?" partitive division.		divided by 5, 250 divided by 5, etc.). **MDD.TC.3c**: Given a known area and a known length dimension, solve a division as finding a missing side length problem by using grid paper to solve (with or without darkened lines around squares of one hundred). **MDD.TC.3d**: Connect the partial quotients recorded in an area model to the partial quotients in written methods. **MDD.TC.3e**: Upon readiness to practice algorithm, practice a limited set with feedback. **MDD.TC.4: Common Instructional Misconceptions** **MDD.TC.4a**: Identify common pitfalls in multi-digit division instruction, including: ● Misconception: Students should transition to the algorithm before students have solid conceptual understanding of expanded algorithms. Correction: Students should spend ample time building a conceptual foundation before transitioning to the algorithm. ● Misconception: Language around how we divide does not need to be precise or match the approach of my curriculum. (e.g., A teacher may say,	

Why Does the Math Make Sense? (MDD.M)	How is This Concept Connected to Other Concepts? (MDD.CC)	What Must I Understand About Teaching This Content? (MDD.TC)	What Must I Understand About Student Reasoning in This Content? (MDD.SR)
		"How many times does a go into b?" to guide a student through their division algorithm even though the curriculum uses partitive language and reasoning.) Correction: The language we use with students needs to be consistent and aligned with the curricular approach to strengthen the connection to concrete and visual models. • Misconceptions: Students can use mnemonic device like DMSB (Dad, Mom, Sister, Brother) to promote steps without understanding. Correction: Mnemonic devices are helpful for memorizing order of steps, but students must be able to understand why each step works and helps them divide.	

MDD.C.1: Understanding Curriculum Design

- **MDD.C.1a**: Identify the vocabulary and definitions a curriculum uses for key concepts and procedures:
  - Regrouping when the greatest place in the dividend must be regrouped to allow division to continue (e.g., regrouping, unbundling, etc.)
  - Division methods (e.g., partial quotients)
  - Using partial quotients to work up to a quotient (e.g., the build up method, etc.)
- **MDD.C.1b**: Identify real-life contexts used consistently with key concepts and procedures:
  - Division (e.g., rows in a theatre, buses of students, etc.)
- **MDD.C.1c**: Identify which visual representations a curriculum uses to represent key concepts and procedures:
  - Division (e.g., arrays, area models, place value disks, place value blocks, etc.)
- **MDD.C.1d**: Identify the language, structures, and symbolic notation a curriculum uses to emphasize the following key understandings:
  - Division
    - Missing Side Length
      - What question begins the division process to find the missing side length?
      - How is the subtraction recorded that indicates the size of the rectangle with the missing side length has now been reduced?
      - Is grid paper used to support understanding?
    - How Many Units in One Group
      - What question begins the division process to find out how many units in one group?
      - How does the curriculum record the partial quotients (with the partial quotients listed moving downward on the right or stacked in the quotient similar to the standard algorithm)?
    - How Many Groups
      - What question begins the division process to find out how many groups?
      - How does the curriculum record the partial quotients (with the partial quotients listed moving downward on the right or stacked in the quotient similar to the standard algorithm)?

MDD.C.2: Task Analysis

MDD.C.2a: Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

Grade 3:

- We can think of and rewrite division problems as multiplication with a missing factor.
- We can understand division as finding the number of units in one group, finding the number of groups, or finding a missing side length.

Grade 4:

- We can use base ten blocks and/or place value charts to enact multi-digit by single-digit division as finding the number of units in each group.
- We can solve "finding the number of units in each group" partial quotients methods by subtracting out groups multiplied by base ten units or more efficiently grouped units.
- We can use base ten blocks to demonstrate multi-digit by single-digit division as finding a missing side length.
- With division as a missing side length and partial quotient, we can use underestimates to build up to the final quotient.
- We can relate calculations used to find the missing side lengths in an area model to the calculations used to find the partial quotients shown in written methods.

Grade 5:

- We can describe division by 10 as moving digits one place to the right.
- With division as a missing side length and partial quotient, we can use underestimates to build up to the final quotient.

MDD.C.3: Coherence

MDD.C.3a: Determine where and how a given curriculum addresses the following standards in grades 4-5 and plan coherent connections in area models or other representations such as place value disks used in base ten multiplication:

- 4.NBT.B.5: Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

  5.NBT.B.5: Fluently multiply multi-digit whole numbers using the standard algorithm.

Content Focus Area:

Fractions: Connecting Visual Representations to Abstract Reasoning

The following content focus area outlines key content to promote understanding fractions first and foremost as numbers. Traditional instruction has not emphasized fractions as numbers, leading to fractions themselves not being thought of as numbers with magnitude and fraction operations being perceived as a series of steps divorced from meaning. Recent research has solidified the number line as the most powerful representation in promoting stronger conceptual understanding of fractions as numbers — fraction instruction should consistently leverage number lines to underscore the meaning of fractions and why operations with fractions make sense.

Fractions as Numbers (FN)

The first fraction competency describes the foundation necessary for strong instruction with fractions as numbers. The recommendations below outline much of the content knowledge and pedagogical content knowledge needed to establish a strong foundation. With that in mind, teachers should take the competencies below and bring them to life in their own classrooms through giving students the opportunity to make sense of fractions by connecting visual and symbolic representations. Teachers should consistently engage students around two central themes: 1) specifying the whole and 2) establishing what is meant by equal parts (CCSS Writing Team, 2019).

Focus Standards
3.NF.A.1: Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*. **3.NF.A.2**: Understand a fraction as a number on the number line; represent fractions on a number line diagram. **3.NF.A.2a**: Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line. **3.NF.A.2a:** Represent a fraction *a*/*b* on a number line diagram by marking off a lengths 1/*b* from 0. Recognize that the resulting interval has size *a*/*b* and that its endpoint locates the number *a*/*b* on the number line.

Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category
3.MD.B.4: Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot where the horizontal scale is marked off in appropriate units — whole numbers, halves, or quarters. **3.G.A.2:** Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.

Why Does the Math Make Sense? (FN.M)	How is This Concept Connected to Other Concepts? (FN.CC)	What Must I Understand About Teaching This Content? (FN.TC)	What Must I Understand About Student Reasoning in This Content? (FN.SR)

FN.M.1: Understanding the Concept Progression

Grade 3:

FN.M.1a: Define a fraction A/B to mean A parts, each of size 1/b, of the unit amount through the use of paper strips, number lines, and tape diagrams (Beckmann, 2011, pp. 48 - 49).[1]

FN.M.1b: Establish the unit fraction as the building block that takes b copies to make the whole through concrete application (folding paper, using tiles, etc.) (Fuson, 2019).

FN.M.1c: Understand the numerator tells us the count of unit fractions that make up the fraction and the denominator determines the whole; use that definition to partition wholes into the unit amount (Beckmann, 2011, p. 49).

FN.M.1d: Demonstrate that fractions are numbers that live on the number line as well as lengths on a number line with linear representations tied to location on a number line.

FN.CC.1: Connections to Prior Learning

FN.CC.1a: Decompose shapes/quantities into halves, fourths (grade 1), and thirds (grade 2) (CCSS Writing Team, 2019, p. 134).[3]

FN.CC.1b: Use words to describe equal shares (one third, one half, etc.).

FN.CC.1c: Explain early elementary measurement concepts such as:

- A number line is like an infinitely long ruler.
- A number on a number line tells its distance from 0 and the length between that number and 0 (National Research Council, 2009, p. 53).
- When our length of a measurement unit decreases, then more of that

FN.TC.1: Visual Diagrams and Representations

FN.TC.1a: Understand the value of the tape diagram as a diagram that:

- may be familiar from work with whole numbers (CCSS Writing Team, 2019, p. 135).
- is less complex geometrically than area models (CCSS Writing Team, 2019, p. 135).
- transfers easily to the number line as a length model (CCSS Writing Team, 2019, p. 135).
- will be used to represent equivalence and operations with fractions as well.

FN.TC.1b: In learning about length measurement, they develop understandings that they will use with number line diagrams such as:

- Length-unit iteration; no gaps or overlaps between successive length-units (CCSS Writing Team, 2019, p. 135)
- Accumulation of length-units to make the total length (CCSS Writing Team,

FN.SR.1: Student Misconceptions and Difficulties

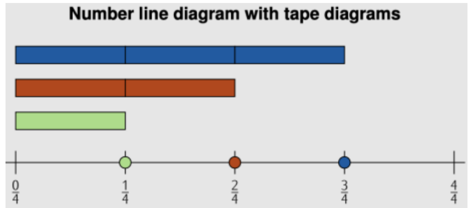
FN.SR.1a: Recognize common misconceptions related to defining fractions such as:

- Misconception: Students see fractions as two distinct wholes instead of a single entity (Frank, 2011, p. 16).
- Misconception: Students believe fractions are only part/whole numbers; in that reasoning a number like 5/3 could not exist because if the whole is partitioned into thirds then there couldn't be 5 pieces (Fazio & Siegler, 2011, p. 10).
- Misconception: Students believe that each fraction has a unique successor (which prevents students from seeing that there are infinite fractions between ⅘ and 1 whereas they know and can count the

[1] 3.NF.A.1: Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*.

[3] 1.G.A.3: Partition circles and rectangles into two and four equal shares, describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares. 2.G.A.3: Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Why Does the Math Make Sense? (FN.M)



(CCSS Writing Team, 2019, p. 137)

FN.M.1e: Demonstrate that fractions can also describe the shaded area of a region determined to be the whole in area representations.

FN.M.1f: In all future fraction content, attend to two important topics:

- Specifying the whole[2]
- Explaining what is meant by "equal parts"

(CCSS Writing Team, 2019, p. 140)

How is This Concept Connected to Other Concepts? (FN.CC)

measurement unit would be needed to cover the same length (National Research Council, 2009, p. 53).

FN.CC.1d: Connect understanding of fractions to understanding of whole numbers in that fractions are composed of units just as a whole numbers are composed of ones (CCSS Writing Team, 2019, p. 135).

FN.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Module

FN.CC.2a: Connect understanding of fractions to measurement by using fraction strips and/or rulers to measure various items so students understand fractions as useful numbers to describe more precise measurements than whole numbers (Fazio & Siegler, 2011, p. 10).[4]

FN.CC.2b: Partition a shape into parts with equal areas in more than one way. Describe the resulting part

What Must I Understand About Teaching This Content? (FN.TC)

2019, p. 135)

- Alignment of zero-point (CCSS Writing Team, 2019, p. 135)
- Meaning of numerals on the ruler; the numerals indicate the number of length units so far (CCSS Writing Team, 2019, p. 135)
- Connecting measurement with physical units and with a ruler (CCSS Writing Team, 2019, p. 135)

FN.TC.1c: Understand the value of the number line as a diagram that:

- illustrates fractions as lengths and individual numbers on number lines (McCallum, 2018).
- expresses fractions as numbers with magnitude (Fazio & Siegler, 2011, p. 10).
- can be used to order numbers (Van de Walle et al., 2013, p. 294).
- extends prior knowledge of whole number concepts rooted in counting by whole number units (1s, 2s, 10s, etc.) to a new unit, 1/b.
- normalizes fractions greater than 1 as logical extensions of the unit fraction

What Must I Understand About Student Reasoning in This Content? (FN.SR)

numbers between 7 and 10) (Fazio & Siegler, 2011, p. 10).

FN.SR.1b: Recognize common misconceptions related to visualizing and representing fractions such as:

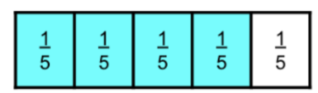
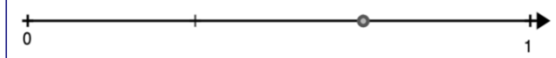
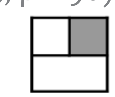
- Misconception: Students believe the whole with 3 parts shaded out of 5 represents the fraction 3/2 because they are not seeing the 3 shaded parts as embedded in a whole composed of fifths (Fuson, 2019).

- Misconception: Students believe that ¾ is shown by any whole decomposed into 4 pieces (regardless as to whether they are equal sized pieces) with 3 pieces shaded (Van de Walle et al., 2013, p. 292).

- Misconception: Students count the number of tick marks and not the number of spaces (e.g.,

[2] Different curricula use different terminology to describe the whole. Other common versions include unit amount or reference amount. For consistency, we use the term "whole" throughout.

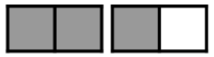
[4] 3.MD.B.4: Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units — whole numbers, halves, or quarters.

Why Does the Math Make Sense? (FN.M)	How is This Concept Connected to Other Concepts? (FN.CC)	What Must I Understand About Teaching This Content? (FN.TC)	What Must I Understand About Student Reasoning in This Content? (FN.SR)
	as a unit fraction of the area of the whole shape.[5] **FN.CC.3: Connections to Future Learning** **FN.CC.3a**: Extend the understanding of fractions as being the totality of m parts when the whole is partitioned into n equal parts to also being interpreted as "the number obtained when "m is divided by n," where the last phrase must be carefully explained with the help of the number line" in grades 5 and 6 (Wu, 2011, p. 3).[6] **FN.CC.3b**: Recognize that the division interpretation of fractions will be the foundation for concepts of percent, ratio, and rate (Wu, 2011, p. 3).[7]	1/b. • extends to encompass understanding and ordering negative fractions as well (Fazio & Siegler, 2011, p. 11). • has an unambiguous whole which is the length of the interval from 0 to 1 (whereas area models require that the whole be defined since it is otherwise ambiguous). **FN.TC.1c**: Recognize tape diagrams written with the unit fractions in each part conceptually reinforce that a fraction a/b represents copies of the unit fraction 1/b (e.g., the fraction represented below is ⅘ because it represents 4 copies of ⅕).  **FN.TC.2: Recommended Instructional Competencies** **FN.TC.2a**: Select and sequence student work that could highlight an important mathematical connection (such as the	assuming the fraction below shows ¾).  • Misconception: Students represent fractions on a number line by order of denominator as if they were whole numbers (e.g., 0, ½, ⅓, ¼, ⅕). • Misconception: Students create unequal spaces when constructing visual models. • Misconception: Students fail to visualize or create equal parts in visual models where additional partitioning lines are needed to create equal parts such as the fraction below (Van de Walle et al., 2013, p. 296).  • Misconception: Students misinterpret the whole (e.g., if the whole is the area of one

[5] 3.G.A.2: Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.

[6] 5.NF.B.3: Interpret a fraction as division of the numerator by the denominator (a/b = a ÷ b). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers (e.g., by using visual fraction models or equations to represent the problem).

[7] 6.RP.A.2: Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship.

Why Does the Math Make Sense? (FN.M)	How is This Concept Connected to Other Concepts? (FN.CC)	What Must I Understand About Teaching This Content? (FN.TC)	What Must I Understand About Student Reasoning in This Content? (FN.SR)
		relationship between the magnitude of the denominator and the size of the denominator, etc.). **FN.TC.2b:** Pose purposeful questions that invite students to make important mathematical connections among different representations of fractions. **FN.TC.2c:** Make the mathematics of the lesson explicit through the use of explanations, representations, tasks, and/or examples. **FN.TC.2d:** Recognize the language demands of a given task: ● Reading ● Writing ● Listening ● Speaking ● Representing ● Interacting **FN.TC.2e:** Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development. **FN.TC.2f:** Adapt common real-life contexts for fractions to connect to student prior knowledge and experience.	rectangle, then the fraction shown is 3/2 whereas if the whole is the area of both rectangles the fraction shown is ¾). 

Why Does the Math Make Sense? (FN.M)	How is This Concept Connected to Other Concepts? (FN.CC)	What Must I Understand About Teaching This Content? (FN.TC)	What Must I Understand About Student Reasoning in This Content? (FN.SR)
		FN.TC.3: Recommended Tasks for Promoting Understanding **FN.TC.3a**: Generate fractions from unit fractions.[8] **FN.TC.3b**: Partition a given whole into unit fractions. **FN.TC.3c**: Partition length models (tape diagrams, number lines) showing one whole into different sized unit fractions and explain how the number of copies needed to make the whole changes. **FN.TC.3d**: Draw length models to represent fractions and explain why the representations fit the definition of a fraction (CCSS Writing Team, 2019, p. 135).[9] **FN.TC.3e**: Locate fractions on a number line given only the position of 0 and another fraction with a like denominator (starting with tick marks in place and eventually without tick marks). **FN.TC.3f**: Determine the whole or the unit amount associated with a fraction given a representation.	

[8] 3.NF.A.2a: Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line.

[9] 3.NF.A.2b: Represent a fraction a/b on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Why Does the Math Make Sense? (FN.M)	How is This Concept Connected to Other Concepts? (FN.CC)	What Must I Understand About Teaching This Content? (FN.TC)	What Must I Understand About Student Reasoning in This Content? (FN.SR)
		***FN.TC.3g**: Find fractional amounts of an object or collection and justify with reasoning (Beckmann, 2011, p. 90).*[10] **FN.TC.4: Common Instructional Misconceptions** **FN.TC.4a:** Understand common language that leads to misconceptions such as: ● Imprecise language: Fractions are "part of a whole." Rationale: This language alone fails to convey the essential meaning of fractions as numbers with magnitude and reinforces misconceptions that fractions are only values between 0 and 1 (Fazio & Siegler, 2011, p. 10). ● Imprecise language: Fractions like 2/3 should be read as "2 over 3" or "2 out of 3." Rationale: These statements fail to underscore the important idea of the numerator as the count and denominator as the unit (Van de Walle et al., 2013, p. 292). ● Imprecise language: Fractions greater than one are "improper fractions." Rationale: Though this term is widespread, it may lead to confusion	

[10] According to the CCSSM, fractions of a set or collection should not come until late grade 4. For this reason, there is not significant attention devoted to this concept in this section.

Why Does the Math Make Sense? (FN.M)	How is This Concept Connected to Other Concepts? (FN.CC)	What Must I Understand About Teaching This Content? (FN.TC)	What Must I Understand About Student Reasoning in This Content? (FN.SR)
		about whether fractions greater than 1 can be expressed in form a/b (Wu, 2011, p. 8). ● Imprecise language: "The whole is the rectangle." Rationale: Describing a whole as a shape instead of the *area* of a shape leads to: ○ disassociation from fractions as numbers — which is crucial to their understanding of fraction operations (Wu, 2011, p. 8). ● Lack of precision in language that will become a barrier when trying to see why "equal parts" does not mean the same thing as "same size and shape" (Wu, 2011, p. 8).	

Equivalent Fractions (EF)

This competency extends the idea that fractions are numbers that live on the number line to explore fractions that live at the same location on the number line. Students initially reason with physical models (paper strips) and visual representations (number lines and tape diagrams) before formalizing their knowledge through symbolic reasoning. Teachers should look for opportunities to elicit different representations from students and pose questions that help students connect the representations. Suggested routines to support reasoning are the 5 Practices and Notice and Wonder.

Focus Standards
3.NF.3: Develop understanding of fractions as numbers. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.) **3.NF.A.3a:** Understand two fractions as equivalent (equal) if they are the same size or the same point on a number line. **3.NF.A.3b:** Recognize and generate simple equivalent fractions (e.g., 1/2 = 2/4, 4/6 = ⅔). Explain why the fractions are equivalent (e.g., by using a visual fraction model). 3.NF.A.3c: Express whole numbers as fractions and recognize fractions that are equivalent to whole numbers. **4.NF.A.1:** Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category
4.NF.2: Extend understanding of fraction equivalence and ordering. Compare two fractions with different numerators and different denominators (e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as ½). Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions (e.g., by using a visual fraction model). (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.) **4.NF.5:** Understand decimal notation for fractions, and compare decimal fractions. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. For example, express 3/10 as 30/100 and add 3/10 + 4/100 = 34/100. (Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.) (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.) **5.NF.1:** Use equivalent fractions as a strategy to add and subtract fractions. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)

EF.M.1: Understanding the Concept Progression

Grade 3:

EF.M.1a: Recognize that equivalent fractions define the same number which can be expressed as a number on a number line, length, or area covered from the same-size whole.[1]

EF.M.1b: Identify whether fractions are equivalent through creating or interpreting visual representations such as area models, tape diagrams, and number lines.[2]

EF.M.1c: Identify fractions equivalent to whole numbers through visual representations.[3]

Grade 4:

EF.M.1d: Given a fraction, use area models, number lines, and/or tape diagrams to explain why multiplying the numerator and denominator by the same number, n, produces an equivalent fraction with n times as many total parts and n times as many shaded parts.[4]

EF.CC.1: Connections to Prior Learning

EF.CC.1a: Connect new learning to earlier conceptions of equivalence as defining the same number (e.g., 5 x 2 is equivalent to 2 x 5) (CCSS Writing Team, 2019, p. 16).[5]

**See prior module on Fractions as Numbers to understand prior knowledge of fractions.*

EF.CC.2 Connections to Other Relevant 3rd - 5th Grade Level Standards to this Module

EF.CC.2a: Connect concepts of equivalence to within grade level work with comparison strategies based on equivalence such as:

- rewriting fractions with common denominators
- rewriting fractions with common numerators
- identifying a fraction equivalent to ½ with a given denominator to use it as a benchmark

EF.CC.2b: Connect concepts of equivalence

EF.TC.1: Visual Diagrams and Representations

EF.TC.1a: Determine which visual representations are most effective in concepts of equivalence:

- Number lines show equivalence well and reinforce that equivalent fractions are the same number on the number line.
- Number lines are generalizable to all rational numbers.
- Length-based representations such as number lines and tape diagrams translate well to fraction operations and reinforce that equivalent fractions are numbers that live at the same place on the number line.

EF.SR.1: Student Misconceptions

EF.SR.1a: Recognize foundational student misconceptions about fractions, including:

- Misconception: Student sees as 2 distinct whole numbers instead of a single entity (Fazio & Siegler, 2011, p. 7).
- Misconception: Numbers within a fraction can be operated on and understood with the same knowledge as whole numbers (e.g., students apply this reasoning to add numerators and add the denominators) (Fazio & Siegler, 2011, p. 6).

EF.SR.1b: Misconception: The equal sign means "the answer is" instead of seeing it as a statement of equivalent (e.g., believing 3 x 4 = 2 x 6 must not be true because the number 12 must follow the equal sign).

[1] 3.NF.A.3a: Understand two fractions as equivalent (equal) if they are the same size or the same point on a number line.

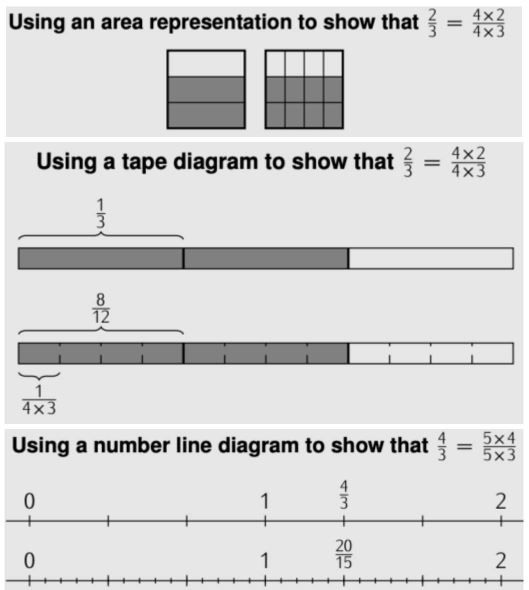
[2] 3.NF.A.3b: Recognize and generate simple equivalent fractions (e.g., 1/2 = 2/4, 4/6 = ⅔). Explain why the fractions are equivalent (e.g., by using a visual fraction model).

[3] 3.NF.A.3c: Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

[4] 4.NF.A.1 Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

[5] 1.OA.D.7: Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 - 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)



(CCSS Writing Team, 2019, p. 143)

EF.M.1e: Formalize written strategies of equivalence to prove the relationship between the original and the equivalent fraction where each unit fraction is further partitioned by a factor of n (CCSS Writing Team, 2019, p. 135).

$$\frac{a\times n}{b\times n} = \frac{an}{bn}$$

EF.M.1f: Use visual representations to show why a fraction such as ¾ is equivalent to 15/20 by explaining how each unit fraction is partitioned into 5 equal parts which yields 5 times as many total

to within grade level work with adding and subtracting fractions such as:

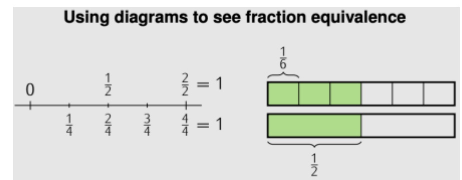
- rewriting fractions greater than 1 as mixed numbers and vice versa using reasoning involving decomposition of mixed numbers as the sum of a whole number and fraction and using equivalent fractions to rewrite the whole number

EF.CC.3: Connections to Future Learning

EF.CC.3a: Extend concepts of equivalence to future work with fraction algorithms based on equivalence to find common units such as:

- addition and subtraction of fractions
- the fraction division algorithm based on common denominators

EF.CC.3b: Explain how equivalent ratios are conceptually similar to and different from equivalent fractions.[6]



(CCSS Writing Team, 2019, p. 143)

EF.TC.1b: Determine which visual representations are common but not as effective in concepts of equivalence:

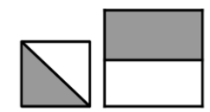
- Circles are not as effective as rectangles because rectangles translate well to number lines/length models and circles can be harder to partition into odd numbered shares (Wu, 2011, p.2).
- Area models become more difficult to represent with fractions such as 17/4 (Wu, 2011, p.20).

EF.TC.2: Symbolic Representations

EF.TC.2a: Recognize the importance of accurate symbolic notation in multiplying numerator and denominator by n since this precedes learning of algorithms for operations

EF.SR.1c: Recognize common student misconceptions in understanding equivalence with visual models such as:

- Misconception: Students draw conclusions about the equivalence of fractions that do not have the same size wholes, such as below (Van de Walle et al., 2013, p. 296).



- Misconception: Students draw conclusions about the equivalence of fractions that are not represented with equal parts (Van de Walle et al., 2013, p. 296).

- Misconception: Students draw conclusions about area models of fractions with the same size whole such as believing that equal parts must mean parts of

[6] 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems (e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations).

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)

parts in the whole and 5 times as many shaded parts (Beckmann, 2011, p. 60).

EF.M.1g: Use symbolic reasoning and formal explanations to explain why multiplying the denominators of two fractions always produces a common denominator (Beckmann, 2011, p. 62).

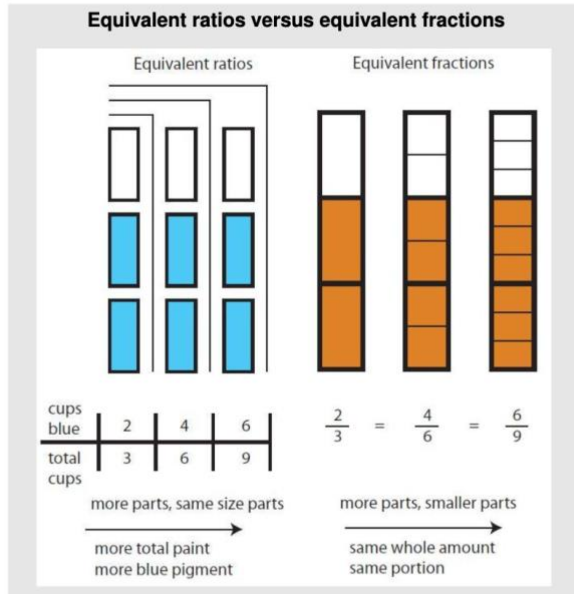
EF.M.1h: *Use symbolic reasoning and formal explanations to show how a fraction not in simplest form can be rewritten in simplest form through a series of divisions of both numerator and denominator by the same non-zero and non-1 whole numbers, n (Beckmann, 2011, p. 63).*

EF.M.1i: Write fractions with denominators of 10 and 100 as equivalent fractions and explain the utility of doing so within our money system (seeing different denominations like dimes as equivalent to 10 cents) and our base ten system (rewriting fractions as decimal fractions or decimals enables us to more easily apply our base ten strategies and algorithms) (CCSS Writing Team, 2019, p. 148).

EF.M.2: Additional Content Knowledge for Teachers

EF.M.2a: Use formal explanations to explain why the following is true for any whole numbers for a and non-zero whole numbers for b and n:

$$\frac{a \times n}{b \times n} = \frac{an}{bn}$$



(CCSS Writing Team, 2019, p. 158)

with fractions (especially multiplication):

$$\frac{a \times n}{b \times n} = \frac{an}{bn}$$

EF.TC.2b: Describe why a relationship of equivalency can be written using multiplication or division.

$$\frac{4 \times 2}{4 \times 3} = \frac{2}{3} \qquad \frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

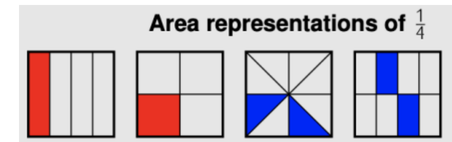
EF.TC.2c: *Use symbolic reasoning and formal explanations to show how a fraction not in simplest form can be rewritten in simplest form through a series of divisions of both numerator and denominator by the same non-zero and non-1 whole numbers, n (Beckmann, 2011, p. 63).*

EF.TC.3: Recommended Instructional Competencies

EF.TC.3a: Make the mathematics of the lesson explicit through the use of explanations, representations, tasks, and/or examples.

EF.TC.3b: Address a common misconception by engaging students in analysis of a task that illustrates the misconception.

the same size and shape (Wu, 2011, p. 8).



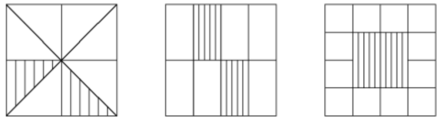
(CCSS Writing Team, 2019, p. 140)

EF.SR.1d: Recognize common student misconceptions about reasoning about equivalence with symbolic notation, including:

- Misconception: Students write whole numbers as fractions by including the whole number in numerator and denominator (e.g., assuming that 6/6 describes the number 6).
- Misconception: Students assume 4/6 and 6/9 cannot be equivalent because there is no whole number, n, that can be multiplied or divided by the numerator and denominator of 4/6 to produce 6/9.
- Misconception: Given two fractions with unlike denominators, students find a common denominator and

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)
EF.M.2b: Explain why any given fraction has infinitely many equivalent fractions (Beckmann, 2011, p. 60). **EF.M.2c**: Justify the cross-multiplication algorithm as "a logical consequence of rewriting the two fractions a/b and d/c as two fractions with the same denominator and then examining their numerators" using visuals and formal symbolic reasoning (Wu, 2011, p. 24). **EF.M.2d**: Understand because every location on a number line corresponds to a number in base ten, it must be possible to represent fractions in base ten as decimals (Beckmann, 2011, p. 53).		**EF.TC.3c**: Recognize the language demands of a given task: • Reading • Writing • Listening • Speaking • Representing • Interacting **EF.TC.3d**: Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development. **EF.TC.3e**: Adapt common real-life contexts for equivalent fractions to connect to student prior knowledge and experience. **EF.TC.4: Recommended Tasks for Promoting Understanding** **EF.TC.4a**: Establish that equivalent fractions describe the same number on the number line by overlaying tape diagrams on number lines to underscore fractions as describing length as well as a point on the number line.	rewrite the fractions without adjusting the numerators. **EF.SR.2: Alternative Methods of Reasoning** **EF.SR.2a**: Recognize that students may learn to cross multiply with limited understanding of why it works and help them draw connections to common denominator comparison (Wu, 2011, p. 24)

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)
		EF.TC.4b: Given a fraction, use math drawings and number lines to explain why multiplying the numerator and denominator by the same number produces an equivalent fraction. **EF.TC.4c:** Generalize that the relationship between further partitioning the unit fractions in a visual model by a factor of n will result in a whole with n times as many total parts and n times as many shaded parts. **EF.TC.4d**: Use visual representations to show why a fraction such as ¾ is equivalent to 15/20 by explaining how each unit fraction is further partitioned into 5 equal parts resulting in 5 times as many total parts in the whole and 5 times as many shaded parts (Beckmann, 2011, p. 60). **EF.TC.4e**: Generate equivalent fractions by multiplying or dividing the numerator and denominator by a non-0 whole number, n. **EF.TC.4f**: Use visual representations to show how two fractions with unlike denominators can be further partitioned (using the other fraction's denominator to further partition each	

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)
		unit fraction) to find common denominators (Beckmann, 2011, p. 62). **EF.TC.4f:** Use formal explanations to explain why multiplying the denominators of two fractions always produces a common denominator (Beckmann, 2011, p. 62). **EF.TC.4g:** Place fractions with unlike denominators on the same number line. **EF.TC.4h:** Use symbolic reasoning and formal explanations to explain why multiplying the denominators of two fractions always produces a common denominator (Beckmann, 2011, p. 62). **EF.TC.4i:** Use different partitionings of the same size whole to reinforce that regions determined to have the same area but not the same shape can still represent equivalent fractions. Such as:  (Wu, 2011, p. 8) **EF.TC.4j:** Reason whether two fractions are equivalent if they each represent the same area relative to two different-sized wholes.	

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)
		EF.TC.4k: *Determine whether a fraction is in simplest form if there is no whole number other than 1 that divides both numerator and denominator evenly (Beckmann, 2011, p. 62).* **EF.TC.4l**: *Use visual representations to show how a fraction not in simplest form can be redrawn by partitioning the whole into fewer equal parts (Beckmann, 2011, p. 63).* **EF.TC.4m**: *Use symbolic reasoning and formal explanations to show how a fraction not in simplest form can be rewritten in simplest form through a series of divisions of both numerator and denominator by the same non-zero and non-1 whole numbers, n (Beckmann, 2011, p. 63).* **EF.TC.4n**: *Generate examples when a fraction in simplest form makes the most sense for an answer and examples of when keeping a fraction in non-simplest form makes more sense.*[7] **EF.TC.5: Common Instructional Misconceptions**	

[7] Simplifying fractions is not explicitly named as part of the CCSSM, therefore these competencies are in italics to indicate they are not required.

Why Does the Math Make Sense? (EF.M)	How is This Concept Connected to Other Concepts? (EF.CC)	What Must I Understand About Teaching This Content? (EF.TC)	What Must I Understand About Student Reasoning in This Content? (EF.SR)
		EF.TC.5a: Understand common language that leads to misconceptions such as: ● Imprecise language: Fractions greater than one are "improper fractions." Rationale: Though this term is widespread, it may lead to confusion about whether fractions greater than 1 can be expressed in form a/b (Wu, 2011, p. 8). ● Imprecise language: The terms "reducing" and "simplifying." Rationale: These terms may imply that a fraction's value has been changed when these processes result in an equivalent fraction (McCallum, 2018).	

Fraction Addition and Subtraction (FAS)

This competency begins the study of operations with fractions. Instruction of fractions operations must aim to make fraction operations as logical as whole number operations. The research base suggests that effective practices include: 1) using contextualized tasks, 2) exploring each operation with a variety of models, especially number lines, 3) providing opportunities to estimate and assess reasonableness, 4) addressing common misconceptions regarding computational procedures (Siegler et al., 2010). With this in mind, teachers should elicit visual representations and encourage students to constantly come back to why fraction operations make sense. Suggested routines to support reasoning are the 5 Practices and Number Talks.

Focus Standards
4.NF.B.3: Understand a fraction a/b with a > 1 as a sum of fractions 1/b. **4.NF.B.3a:** Understand addition and subtraction of fractions as joining and separating parts referring to the same whole. **4.NF.B.3b:** Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions (e.g., by using a visual fraction model). Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8. **4.NF.B.3c:** Add and subtract mixed numbers with like denominators (e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction). **4.NF.B.3d:** Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators (e.g., by using visual fraction models and equations to represent the problem). **4.NF.B.4:** Apply and extend previous understandings of multiplication to multiply a fraction by a whole number. **4.NF.B.4a:** Understand a fraction a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4). **4.NF.C.5:** Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100. **5.NF.A.1:** Use equivalent fractions as a strategy to add and subtract fractions. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.) **5.NF.2:** Use equivalent fractions as a strategy to add and subtract fractions. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators (e.g., by using visual fraction models or equations to represent the problem). Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result 2/5 + 1/2 = 3/7 by observing that 3/7 < 1/2.

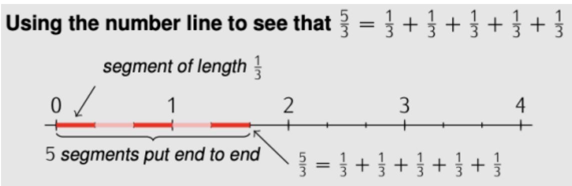
Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category

4.MD.B.4: Represent and interpret data. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.
5.MD.B.2: Represent and interpret data. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.
5.NBT.B.7: Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Why Does the Math Make Sense? (FAS.M)	How is This Concept Connected to Other Concepts? (FAS.CC)	What Must I Understand About Teaching This Content? (FAS.TC)	What Must I Understand About Student Reasoning in This Content? (FAS.SR)
FAS.M.1: Understanding the Concept Progression **Grade 4:** **FAS.M.1a**: Demonstrate addition and subtraction of fractions with like denominators using paper strips, tape diagrams, and number lines. **FAS.M.1b**: Relate addition and subtraction of fractions to addition and subtraction of whole numbers by demonstrating the joining of segments on a number line (defined by intervals the size of the unit fraction with fractions with common denominators and then by intervals the size of whole numbers) (Wu, 2011, p. 24).	**FAS.CC.1: Connections to Prior Learning** **FAS.CC.1a**: Connect addition and subtraction of fractions to addition and subtraction of whole numbers by demonstrating the joining of segments on a number line (Wu, 2011, p. 24). **FAS.CC.1b**: Connect concepts of addition and subtraction of units such as tens or hundreds to addition and subtraction of fractions with common units. **FAS.CC.1c**: Describe how the same mathematical concepts are used to regroup when needed in subtraction of whole numbers as with subtraction of mixed numbers.	**FAS.TC.1: Visual Diagrams and Representations** **FAS.TC.1a**: Recognize the research base that supports students using visual representations (number lines, area models, and tape diagrams) to explain why procedures for computing with fractions work (Siegler et al, 2010, p.28).	**FAS.SR.1: Student Misconceptions** **FAS.RS.1a**: Recognize common errors in student visual representations of adding and subtracting fractions, including: • Misconception: Students subtract by crossing out pieces and changing the size of the whole (e.g., $\frac{2}{3} - \frac{1}{3} = \frac{1}{2}$). 

Why Does the Math Make Sense? (FAS.M)

FAS.M.1c: Write fractions as sums of unit fractions (e.g., $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$) and demonstrate why this is true using a tape diagram or number line (Wu, 2011, p.25).



(CCSS Writing Team, 2019, p. 144)

FAS.M.1d: Record fraction addition and subtraction using unit form such as $\frac{5}{8} - \frac{1}{8}$ as 5 eighths - 1 eighth (Beckmann, 2011, p. 63).

FAS.M.1e: Rewrite a fraction greater than 1 as a mixed number by decomposing the fraction into a sum of a whole number and a number less than 1.

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}.$$

(CCSS Writing Team, 2019, p. 145)

FAS.M.1f: Rewrite mixed numbers as fractions by representing the mixed number as sum of a whole number and fraction and then representing the whole number as an equivalent fraction with the same denominator as the fraction before adding.

How is This Concept Connected to Other Concepts? (FAS.CC)

**See prior sub-categories on Fractions as Numbers to understand prior knowledge of fractions.*

FAS.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Module

FAS.CC.2a: Add and subtract decimal numbers by rewriting them first as decimal fractions and then rewriting the numbers using equivalent decimal fractions with common denominators.

FAS.CC.2b: Use addition and subtraction to answer questions relating to line plots with fractions.

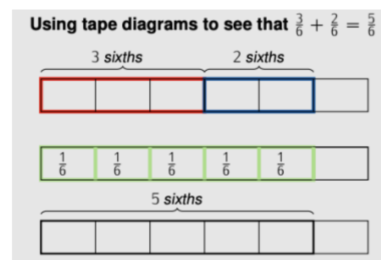
FAS.CC.3: Connections to Future Learning

FAS.CC.3a: Demonstrate how common units are needed to add and subtract quantities, including algebraic terms.[1]

What Must I Understand About Teaching This Content? (FAS.TC)

FAS.TC.1b: Determine which visual representations are effective in showing addition and subtraction of fractions:

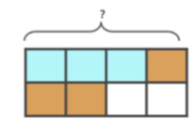
- Number lines are generalizable to all rational numbers and all operations.
- Tape diagrams translate well to fraction operations and can be written with unit fractions inside to reinforce the meaning of the operations based on the individual unit fractions.



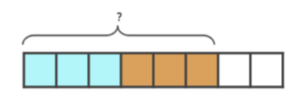
(CCSS Writing Team, 2019, p. 145)

What Must I Understand About Student Reasoning in This Content? (FAS.SR)

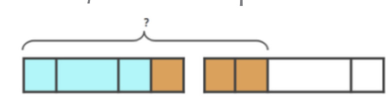
- Misconception: Students add ¾ and ¾ by stacking the wholes to create what appears to be a whole composed of eighths instead of fourths.



- Misconception: Students use a tape diagram without defining the whole so that it appears the sum is 6/8.



- Misconception: Students use unequal parts so that the visual does not reflect the size/relationship of the sum.



[1] 7.EE.A.1: Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

Why Does the Math Make Sense? (FAS.M)	How is This Concept Connected to Other Concepts? (FAS.CC)	What Must I Understand About Teaching This Content? (FAS.TC)	What Must I Understand About Student Reasoning in This Content? (FAS.SR)

$$7\frac{1}{5} = 7 + \frac{1}{5} = \frac{35}{5} + \frac{1}{5} = \frac{36}{5}$$

(CCSS Writing Team, 2019, p. 145)

FAS.M.1g: Write equivalent forms of mixed numbers, including forms that show one whole has been regrouped (e.g., 4 ⅔ = 3 5/3).

FAS.M.1h: Add and subtract mixed numbers using visual models (tape diagrams and number lines).

FAS.M.1i: Add and subtract mixed numbers using symbolic form. Describe why each step makes sense.

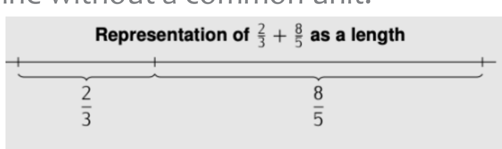
FAS.M.1j: Add and subtract decimal fractions with unlike denominators by rewriting fractions with a denominator of 10 as equivalent fractions with a denominator of 100.

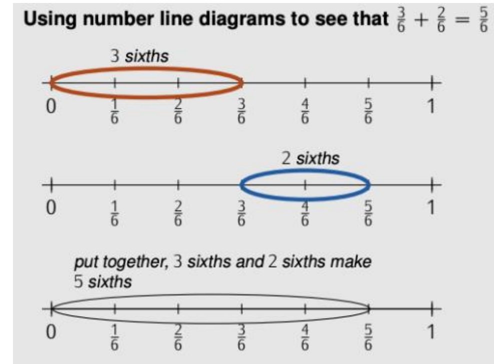
$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

(CCSS Writing Team, 2019, p. 148)

Grade 5:

FAS.M.1k: Demonstrate that the lengths of two fractions with unlike denominators joined together compose a longer length whose length is hard to determine without a common unit.





(CCSS Writing Team, 2019, p. 145)

FAS.TC.1c: Determine which visual representations are common but not as effective for addition and subtraction of fractions:

- Circles are not as effective as rectangles because rectangles translate well to number lines/length models and circles can be harder to partition into odd numbered shares (Wu, 2011, p.2).

FAS.TC.2: Symbolic Diagrams and Representations

FAS.TC.2a: Determine which symbolic representations of addition and subtraction of fractions are effective including:

- Writing fractions in unit form

- Misconception: Students use unequal-sized wholes.

FAS.SR.1b: Recognize common misconceptions related to not seeing a fraction as a single entity but rather two distinct whole numbers (Fazio & Siegler, 2011, p. 7), including:

- Misconception: Students add or subtract the numerator from the numerator and the denominator from the denominator (Fazio & Siegler, 2011, p. 10).

FAS.SR.1c: Recognize the following common procedural misconceptions:

- Misconception: Students find a common denominator without rewriting the numerator to create equivalent fractions (e.g., ½ + ⅔ = ⅙ + 2/6).

FAS.SR.1d: Recognize the following common misconceptions and their connections to misconceptions with whole number addition and subtraction:

Why Does the Math Make Sense? (FAS.M)	How is This Concept Connected to Other Concepts? (FAS.CC)	What Must I Understand About Teaching This Content? (FAS.TC)	What Must I Understand About Student Reasoning in This Content? (FAS.SR)

Why Does the Math Make Sense? (FAS.M)

(Wu, 2011, p. 36)

FAS.M.1l: Estimate the sum or difference of given fractions. After performing computation, check answers for reasonableness.

FAS.M.1m: Replace given fractions with equivalent fractions so that they can be added or subtracted with like denominators in the following scenarios:

- When one denominator divides the other denominator
- When no number except 1 divides both denominators
- When some number divides both denominators

FAS.M.1n: Justify the generalization that a/b + c/d = (ad + bc)/bd for all non-zero whole numbers.

$$\frac{a}{b}+\frac{c}{d}=\frac{a\times d}{b\times d}+\frac{c\times b}{d\times b}=\frac{a\times d+b\times c}{b\times d}.$$

(CCSS Writing Team, 2019, p. 150)

What Must I Understand About Teaching This Content? (FAS.TC)

(e.g., 2 fifths) to emphasize the denominator as the unit and to underscore clear connections to addition and subtraction as whole numbers

- Recording the numerator as the sum of units or as the sum or difference of the numerators over a common denominator

$$\frac{3}{6}+\frac{2}{6}=\overbrace{\frac{1}{6}+\frac{1}{6}+\frac{1}{6}}^{\text{3 sixths}}+\overbrace{\frac{1}{6}+\frac{1}{6}}^{\text{2 sixths}}$$
$$=\frac{\overbrace{1+1+1+1+1}^{\text{3+2 sixths}}}{6}$$
$$=\frac{5}{6}$$

(CCSS Writing Team, 2019, p. 145)

FAS.TC.3: Recommended Instructional Competencies

FAS.TC.3a: Select and sequence student work that could highlight an important mathematical connection (such as the relationship between visual models and the procedural steps, the connections between whole number addition and subtraction and

What Must I Understand About Student Reasoning in This Content? (FAS.SR)

- Misconception: Students do not regroup whole numbers when subtracting mixed numbers from whole numbers (e.g., 4 - 1 ⅔ = 3 ⅔ or 3 ⅓).
- Misconception: Students do not regroup mixed numbers when necessary or "subtracting up" (e.g., 3 ⅛ - 1 6/8 = 2 ⅝).

FAS.SR.2 Student Supports

FAS.SR.2a: Use visual representations to add or subtract fractions with unlike denominators limited to 2, 4, or 8 so that students with unfinished learning begin by only needing to rewrite or further partition one fraction (Van de Walle et al., 2013, p. 319).

FAS.SR.2b: Estimate the sum or difference before computing so students can check their answers for reasonableness (Fazio & Siegler, 2011, p. 14).

FAS.SR.2c: Provide opportunities to demonstrate fraction addition and subtraction on a number line so that students with conceptual gaps benefit from the following strengths of number lines as diagrams:

Why Does the Math Make Sense? (FAS.M)	How is This Concept Connected to Other Concepts? (FAS.CC)	What Must I Understand About Teaching This Content? (FAS.TC)	What Must I Understand About Student Reasoning in This Content? (FAS.SR)
		fraction addition and subtraction, etc.). **FAS.TC.3b**: Pose purposeful questions that invite students to make important mathematical connections among different representations of fractions. **FAS.TC.3c**: Make the mathematics of the lesson explicit through the use of explanations, representations, tasks, and/or examples. **FAS.TC.3d**: Address a common misconception by engaging students in analysis of a task that illustrates the misconception. **FAS.TC.3e**: Recognize the language demands of a given task: • Reading • Writing • Listening • Speaking • Representing • Interacting **FAS.TC.3f:** Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development.	• They express fractions as numbers with magnitude (Fazio & Siegler, 2011, p. 10). • They can be used to order numbers (Van de Walle et al., 2013, p. 294). • They extend prior knowledge of whole number concepts rooted in counting by whole number units (1s, 2s, 10s, etc.) to a new unit, 1/b. • They have an unambiguous whole, which is the length of the interval from 0 to 1 (whereas area models require that the whole be defined since it is otherwise ambiguous). **FAS.SR.2d**: Demonstrate regrouping in addition and subtraction of mixed numbers and ask students to draw connections to whole number addition and subtraction with regrouping.

Why Does the Math Make Sense? (FAS.M)	How is This Concept Connected to Other Concepts? (FAS.CC)	What Must I Understand About Teaching This Content? (FAS.TC)	What Must I Understand About Student Reasoning in This Content? (FAS.SR)
		EF.TC.3g: Adapt common real-life contexts for fraction addition and subtraction to connect to student prior knowledge and experience. **FAS.TC.4: Recommended Tasks for Promoting Understanding** **FAS.TC.4a:** Use a visual representation such as a tape diagram or number line to explain why we do not add the denominators when adding two fractions. **FAS.TC.4b:** Given several visual representations, decide which can be used to show addition and subtraction of fractions (where various representations accurately show the addition or subtraction and those that do not illustrate common misconceptions such as adding fractions that describe different-sized wholes). **FAS.TC.4c:** Sum sets of mixed numbers with like denominators efficiently by looking for fractional parts that sum to 1.	

Why Does the Math Make Sense? (FAS.M)	How is This Concept Connected to Other Concepts? (FAS.CC)	What Must I Understand About Teaching This Content? (FAS.TC)	What Must I Understand About Student Reasoning in This Content? (FAS.SR)
		FAS.TC.5: Common Instructional Misconceptions **FAS.TC.5a:** Identify common pitfalls or limitations with existing strategies for addition and subtraction of fractions, including: • Limitation: The "butterfly method" proceduralizes addition or subtraction of two fractions but is not taught conceptually and cannot be applied to problems such as three addend addition. • Limitation: Converting all mixed numbers to fractions greater than 1, which results in losing a sense of the magnitude of the numbers involved and does not hold the same connections to base ten addition and subtraction as techniques with regrouping.	

What Must I Understand About My Curriculum's Approach to This Content? (FAS.C)

FAS.C.1: Understanding Curriculum Design

- **FAS.C.1a**: Identify the vocabulary and definitions a curriculum uses for key concepts and procedures:
  - Regrouping (e.g., rewriting 2 ¾ as 1 5/4)
  - Common denominators (e.g., common units, like denominators, etc.)
- **FAS.C.1b**: Identify real-life contexts used consistently with key concepts and procedures:
  - Fraction addition and subtraction (e.g., races, pizzas, brownies, etc.)
- **FAS.C.1c**: Identify which visual representations a curriculum uses to represent key concepts and procedures:
  - Fraction addition and subtraction (e.g., number lines, area models, tape diagrams, etc.)
- **FAS.C.1d**: Identify the language, structures, and symbolic notation a curriculum uses to emphasize the following key understandings:
  - Adding and subtracting fractions with like denominators (e.g., Do they use unit form? Do they rewrite the sum or difference in the numerator with the fraction bar extending over a single denominator?)
  - Rewriting mixed numbers as fractions greater than one and vice versa (e.g., Do they consistently show the in-between step of the whole number being rewritten as a fraction greater than one with a like denominator? Do they use number bonds to illustrate the decomposition of the whole and fractional part?)
  - Finding like denominators (e.g., Do they encourage students to find the least common multiple? Do they have an organizational structure students use to find common multiples? Do they write the multiplication needed to find the equivalent fraction as two distinct fractions being multiplied or as a separate multiplication recorded in the numerator and a separate multiplication recorded in the denominator?)
  - Adding mixed numbers (e.g., How do they show what happens when regrouping the sum is required? What different strategies for addition do they encourage?)
  - Subtracting mixed numbers (e.g., How do they show what happens when regrouping the minuend is required? What different strategies for subtraction do they encourage?)

FAS.TC.2: Task Analysis

FAS.TC.2a: Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

- Addition and subtraction of fractions extends addition and subtraction of whole numbers by applying the same concepts with units other than ones.
- We can express fractions as the sum of unit fractions.
- We can visually demonstrate fraction addition and subtraction and use it to explain the procedure for adding and subtracting fractions.
- The lengths of two fractions with unlike denominators joined together compose a longer length whose length is hard to determine without a common unit.

FAS.TC.2b: Select tasks from a curriculum that will surface one of the following misconceptions and plan a curriculum-aligned explanation linking the misunderstanding to a mathematical concept:

- Adding or subtracting the numerator from the numerator and the denominator from the denominator
- Finding a common denominator without rewriting the numerator to create equivalent fractions (e.g., $\frac{1}{2} + \frac{2}{3} = \frac{1}{6} + 2/6$)
- Not regrouping whole numbers when subtracting mixed numbers from whole numbers (e.g., $4 - 1\frac{2}{3} = 3\frac{2}{3}$ or $3\frac{1}{3}$)
- Not regrouping mixed numbers when necessary or "subtracting up" (e.g., $3\frac{1}{8} - 1\ 6/8 = 2\frac{5}{8}$)

FAS.TC.3: Coherence

FAS.TC.3a: Determine where and how a given curriculum addresses the following standards and describe how these standards apply concepts of fraction addition and subtraction:

- 4.MD.B.4: Represent and interpret data. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.
- 5.MD.B.2: Represent and interpret data. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.
- 5.NBT.B.7: Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Fraction Multiplication and Division (FMD)

This competency explores fraction multiplication and division. The content of this section is grounded in the definition of multiplication (M x N) as the number of units (or objects) in M equal groups if there are N units (or objects) in one group where then division can be used to figure out either the number of groups or the number of units in one group. This definition becomes the anchor for the more challenging application of fraction multiplication and division such as thinking of the number of groups in multiplication when the number of groups is less than one (e.g., ⅔ x 5 is "What is ⅔ of a group of 5?") and the "how many groups" interpretation of division (e.g., ⅔ ÷ 7 is "What fraction of a group of ⅔ is 7?").

Given the extensive experience students in grades 3-5 have working with this understanding of whole numbers, that same understanding should be the bedrock of instruction for fraction multiplication and division.

In line with the recommendations from the section on fraction addition and subtraction, strong research-based practice centers around: 1) using contextualized tasks, 2) exploring each operation with a variety of models, especially number lines, 3) providing opportunities to estimate and assess reasonableness, 4) addressing common misconceptions regarding computational procedures (Siegler et al., 2010). Teachers should continue to elicit visual representations and encourage students to constantly come back to why fraction operations make sense. Suggested routines to support reasoning are the 5 Practices and Number Talks.

Focus Standards

4.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

4.NF.B.4a: Understand a fraction a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4).

4.NF.B.4b: Understand a multiple of a/b as a multiple of 1/b, and use this understanding to multiply a fraction by a whole number. For example, use a visual fraction model to express 3 × (2/5) as 6 × (1/5), recognizing this product as 6/5. (In general, n × (a/b) = (n × a)/b.)

5.NF.B.3: Interpret a fraction as division of the numerator by the denominator (a/b = a ÷ b). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers (e.g., by using visual fraction models or equations to represent the problem). For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?

5.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

5.NF.B.4a: Interpret the product (a/b) × q as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations a × q ÷ b. For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = (ac)/(bd).)

5.NF.B.4b: Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

5.NF.B.5: Interpret multiplication as scaling (resizing), by:

5.NF.B.5a: Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

5.NF.B.5b: Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence a/b = (n × a)/(n × b) to the effect of multiplying a/b by 1.

5.NF.B.6: Solve real world problems involving multiplication of fractions and mixed numbers (e.g., by using visual fraction models or equations to represent the problem).

5.NF.B.7: Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

5.NF.B.7a: Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3.

5.NF.B.7b: Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4.

5.NF.B.7c: Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions (e.g., by using visual fraction models and equations to represent the problem). For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?

Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category

3.OA.B.5: Apply properties of operations as strategies to multiply and divide. Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known (Commutative property of multiplication). 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30 (Associative property of multiplication). Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56 (Distributive property).

3.OA.B.6: Understand division as an unknown-factor problem. For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by **8.**

3.NBT.A.3: Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

3.MD.C.7b: Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

5.NBT.B.7: Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Why Does the Math Make Sense? (FMD.M)	How is This Concept Connected to Other Concepts? (FMD.CC)	What Must I Understand About Teaching This Content? (FMD.TC)	What Must I Understand About Student Reasoning in This Content? (FMD.SR)

FMD.M.1: Understanding the Concept Progression

Grade 4:

FMD.M.1a: Write fraction a/b as a sum of a copies of 1/b as well as a product of a and 1/b. Use visual representations (tape diagrams, number lines, etc.) to illustrate why both expressions represent the same number.

FMD.M.1b: Explain why n x a/b can be rewritten as $\frac{(n \times a)}{b}$ using visual representations (tape diagrams, number lines, etc.).

FMD.M.1c: Rewrite multiplication of a whole number by a mixed number two ways:

- Multiplying a whole number by a fraction greater than one (e.g., 3 x 1 ½ = 3 x 3/2)
- Using the distributive property to multiply both addends by n (e.g., 3 x 1 ½ = (3 x 1) + (3 x ½))

(CCSS Writing Team, 2019, p. 148)

FMD.M.1d: Create an accurate visual representation to show a contextualized problem involving a whole number being multiplied by a fraction.

FMD.M.1e: Create a story context that could be used to describe a whole number being multiplied by a fraction.

Grade 5:

FMD.CC.1: Connections to Prior Learning

**See prior sub-categories on Fractions as Numbers to understand prior knowledge of fractions.*

Grade 3:

FMD.CC.1a: Rewrite division equations as a missing factor multiplication equation.[1]

FMD.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Module

FMD.CC.2a: Extend understanding of the associative property with whole number multiplication (e.g., 4 x 30 is 4 times 3 tens) to fraction multiplication (e.g., 4 x ⅗ is 4 x 3 fifths).

FMD.CC.2b: Extend a rectangular area model with whole number side lengths (e.g., 3 units x 5 units) to become a rectangle with fractional side lengths (e.g., 3 ½ units x 5 ½ units) and explain how the same principles of finding the total unit squares covered will

FMD.TC.1: Visual Diagrams and Representations

FMD.TC.1a: Recognize the research base that supports students using visual representations (number lines, area models, and tape diagrams) to explain why procedures for computing with fractions work (Siegler et al, 2010, p.28).

FMD.TC.2: Symbolic Diagrams and Representations

FMD.TC.2a: Compare the benefits of writing fraction multiplication in the following forms:

$\frac{2\times4}{3\times4} = \frac{8}{12}$ and $\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$

- The form on the left is appropriate if students have not learned fraction multiplication (e.g., when they begin grade 4 work on equivalent fractions). It transfers to dividing numerator and denominator by the same number to find an equivalent

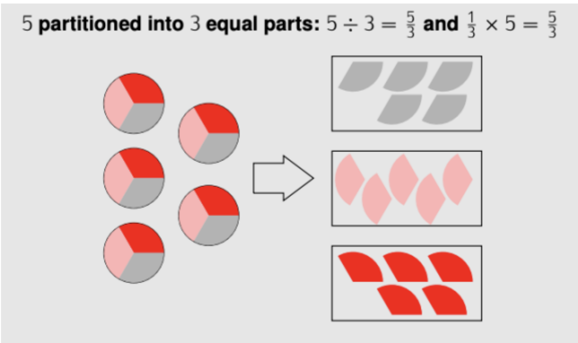
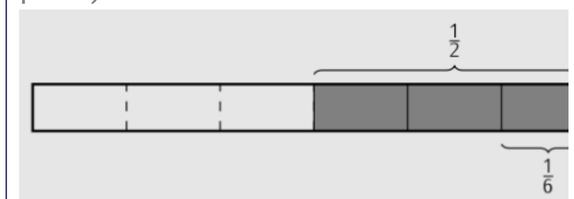
FMD.SR.1: Student Misconceptions

FMD.SR.1a: Recognize the following common misconceptions and the underlying conceptual gaps and plan effective strategies such as using area models with partial products to address them:

- Misconception: Mixed numbers can be multiplied by multiplying the whole numbers and multiplying the fractions (e.g., 2 ½ x 3 1/3 = 6 1/6).
- Misconception: To multiply a whole number by a fraction, multiply the numerator and denominator by the whole number.
- Misconception: Fractions should be rewritten with common denominators before multiplying the numerators only.

FMD.SR.1b: Misconception: When using visual models to solve problems like $\frac{1}{2} \div 3$ =?and partition the shaded ½ into

[1] 3.OA.B.6: Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8*.

Why Does the Math Make Sense? (FMD.M)	How is This Concept Connected to Other Concepts? (FMD.CC)	What Must I Understand About Teaching This Content? (FMD.TC)	What Must I Understand About Student Reasoning in This Content? (FMD.SR)
FMD.M.1f: Explain why $a \div b$ can be understood as $\frac{a}{b}$ or $a \times \frac{1}{b}$ using real-life situations and visual representations demonstrating partitive division.  *If you share 5 objects equally among 3 people, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus, each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object. Because $5 \times \frac{1}{3} = \frac{5}{3}$, each share is $\frac{5}{3}$ of an object.* (CCSS Writing Team, 2019, p.151) **FMD.M.1g:** Visually represent and explain two different approaches that could be used to represent a sharing situation (e.g., in the 5 objects shared equally among 3 people scenario, students might automatically deal out one whole to each person and then partition the remaining two wholes into thirds). **FMD.M.1h:** Create an accurate visual representation to show a contextualized problem involving fractions as sharing. **FMD.M.1i:** Create a story context that could be used to	determine the area. **FMD.CC.2c:** Rewrite multiplication with decimal numbers as multiplication with decimal fractions and find the product. **FMD.CC.2d:** Apply understanding of fractions to reasoning about how many tenths or hundredths are in a given whole number. **FMD.CC.3: Connections to Future Learning** **FMD.CC.3a:** Explain why the algorithm for dividing fractions makes sense. **FMD.CC.3b:** Extend division to find the value of the ratio a:b as a unit rate.[2]	fraction and it will be useful when students learn to find common factors in both the numerator and denominator to see if it is possible to simplify by finding common factors first rather than multiplying. • The form on the right makes it more explicit that we are multiplying ⅔ by 4/4 or 1 whole, which does not change the value of ⅔. **FMD.TC.3: Recommended Instructional Competencies** **FMD.TC.3a:** Pose purposeful questions that invite students to make important mathematical connections among different representations of fraction operations. **FMD.TC.3b:** Make the mathematics of the lesson explicit through the use of explanations, representations, tasks, and/or examples. **FMD.TC.3c:** Address a common misconception by engaging students	third, students may not partition the remaining ½. Without an evenly partitioned whole, students may produce answers like ⅓ (believing that ½ is now the whole) or ¼ (seeing that the whole is now partitioned into 4 parts).  **FMD.SR.1c:** Misconception: when dividing a whole number by a fraction, students put the quotient back with the fraction units. (e.g., When students reason about $3 \div \frac{1}{4} = 12$, they often reason that there are 12 fourths in 3. Though they understand the concept, they struggle to determine whether the answer should be expressed as $\frac{12}{4}$ or 12.)

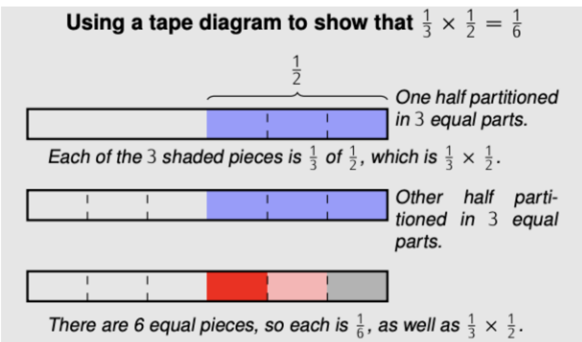
[2] 6.RP.A.2: Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."

Why Does the Math Make Sense? (FMD.M)	How is This Concept Connected to Other Concepts? (FMD.CC)	What Must I Understand About Teaching This Content? (FMD.TC)	What Must I Understand About Student Reasoning in This Content? (FMD.SR)

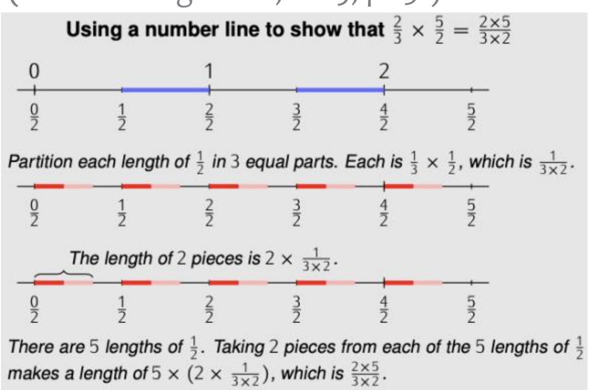
describe fractions as sharing.

FMD.M.1j: Write verbal descriptions of fraction a/b as a share of a whole (e.g., ⅚ can be understood as ⅙ of 5) and connect it to the understanding of a/b as a x 1/b developed through exploring fractions as sharing (CCSS Writing Team, 2019, p.151).

FMD.M.1k: Using tape diagrams and number lines, demonstrate fraction multiplication and explain why $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.



(CCSS Writing Team, 2019, p.151)



in analysis of a task that illustrates the misconception.

FMD.TC.3d: Recognize the language demands of a given task:

- Reading
- Writing
- Listening
- Speaking
- Representing
- Interacting

FMD.TC.3e: Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development.

FMD.TC.3f: Adapt common real-life contexts for fraction multiplication and division to connect to student prior knowledge and experience.

FMD.TC.4: Recommended Tasks for Promoting Understanding

FMD.TC.4a: Given a real world context, explain how fraction multiplication or division could be applied using words and/or visuals (Siegler et al, 2010, p.27).

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<th>Why Does the Math Make Sense?

(FMD.M)</th>
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(FMD.CC)</th>
<th>What Must I Understand About Teaching This Content?
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<th>What Must I Understand About Student Reasoning in This Content?
(FMD.SR)</th>
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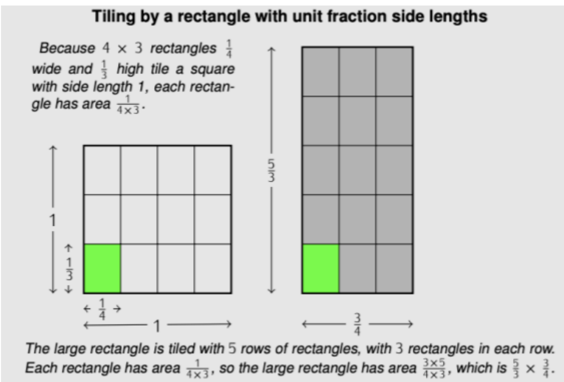
(CCSS Writing Team, 2019, p.152)

FMD.M.1l: Create a story context that could be used to describe multiplication of fractions involving redefining the whole.

FMD.M.1m: Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths.



(CCSS Writing Team, 2019, p.152)

FMD.M.1n: Rewrite equations involving division of fractions as multiplication with a missing factor equations.

- $m \div \frac{1}{n} = ?$ can be rewritten as $? \times \frac{1}{n} = m$
- $\frac{1}{n} \div m = ?$ can be rewritten as $? \times \frac{1}{m} = n$

FMD.M.1o: Using a visual representation, reason about how many unit fractions are needed to make a whole. Extend that reasoning to figure out how many unit fractions are needed to make a different whole number (Wu, 2011, p. 56).

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FMD.TC.4b: Identify which representations can be used to show 4 x ¾ with correct responses including repeated addition of ¾, a visual representation showing ¾ of 4, an area model showing a rectangle that is 4 units long and ¾ units wide, etc.

FMD.TC.4c: Describe a context in which multiplication results in a product smaller than the original number.

FMD.TC.4d: Describe a context in which division results in a quotient larger than the original number.

FMD.TC.5: Common Instructional Misconceptions

FMD.TC.5a: Identify common pitfalls or limitations with existing strategies for multiplication of fractions, including:

- Misconception: The simplest way to learn fraction multiplication is to learn the procedures. Correction: Students who do not understand why procedures work are not likely to carry them out consistently or

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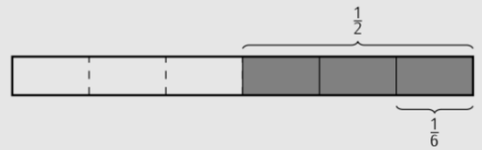
Why Does the Math Make Sense? (FMD.M)	How is This Concept Connected to Other Concepts? (FMD.CC)	What Must I Understand About Teaching This Content? (FMD.TC)	What Must I Understand About Student Reasoning in This Content? (FMD.SR)

FMD.M.1p: Create a story context that could be used to describe division of a fraction by a whole number and vice versa.

FMD.M.1q: Create visual representations that could be used to solve for division of a fraction by a whole number and vice versa.

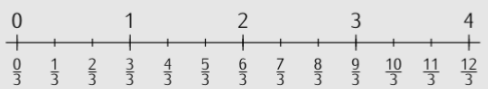
Division of a unit fraction by a whole number: $\frac{1}{2} \div 3$



Reasoning with a tape diagram using the sharing interpretation of division: the tape is the whole and the shaded length is $\frac{1}{2}$ of the whole. If the shaded length is partitioned into 3 equal parts, then 2×3 of those parts compose the whole, so $\frac{1}{2} \div 3 = \frac{1}{2\times 3} = \frac{1}{6}$.

(CCSS Writing Team, 2019, p.153)

Division of a whole number by a unit fraction: $4 \div \frac{1}{3}$



Reasoning on a number line using the measurement interpretation of division: there are 3 parts of length $\frac{1}{3}$ in the unit interval, therefore there are 4×3 parts of length $\frac{1}{3}$ in the interval from 0 to 4, so the number of times $\frac{1}{3}$ goes into 4 is 12, that is $4 \div \frac{1}{3} = 4 \times 3 = 12$.

(CCSS Writing Team, 2019, p.153)

FMD.M.1r: Predict whether the product will be smaller than, equal to, or larger than the initial number based on the size of the following factor relative to 1.

appropriately.

- Misconception: We can use key words such as "of" or "each" to indicate a word problem should be solved with multiplication. Correction: This approach often does not work and it detracts from students learning to solve word problems through careful reasoning.

FMD.TC.5b: Identify common pitfalls or limitations with existing strategies for division of fractions, including:

- Misconception: The simplest way to learn fraction division is to learn the procedures. Correction: Students who do not understand why procedures work are not likely to carry them out consistently or appropriately.

FMD.TC.5c: Identify statements that are common but imprecise and explain why they are inaccurate:

- Imprecise language: "Multiplication makes numbers bigger." Rationale: This

Why Does the Math Make Sense? (FMD.M)	How is This Concept Connected to Other Concepts? (FMD.CC)	What Must I Understand About Teaching This Content? (FMD.TC)	What Must I Understand About Student Reasoning in This Content? (FMD.SR)
FMD.M.1s: Predict whether a quotient will be smaller than, equal to, or larger than the dividend based on the divisor's size relative to the dividend. **FMD.M.2: Additional Math Content Knowledge for Teachers** **FMD.M.2a**: Create a story problem involving redefining the whole and a story problem that does not involve redefining the whole. **FMD.M.2b**: Given a division of fractions story problem, identify whether the problem is asking "how many units in one group?" or "how many groups?" **FMD.M.2c**: Using mathematical reasoning, explain why dividing is equivalent to multiplying by the reciprocal. $\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{a}{b} \times \frac{d}{c}$ **FMD.M.2d**: Explain why a fraction division problem can be solved by rewriting fractions with equivalent fractions with common denominators and dividing the numerators (e.g., $\frac{1}{2} \div \frac{1}{6} = \frac{3}{6} \div \frac{1}{6} = 3 \div 1 = 3$).		statement does not hold true when a number is multiplied by a factor less than or equal to 1. • Imprecise language: "Division gives a smaller answer." Rationale: This statement does not hold true when a divisor is smaller than 1.	

What Must I Understand About My Curriculum's Approach to This Content? (FMD.C)

FMD.C.1: Understanding Curriculum Design

- Identify the vocabulary and definitions a curriculum uses for key concepts and procedures:
  - Measurement division
  - Partitive division
- Identify real-life contexts used consistently with key concepts and procedures:
  - Multiplication (e.g., painting a wall, a glass of water, a pan of brownies, etc.)
  - Division (e.g., ribbon, racing, a pan of brownies, etc.)
- Identify which visual representations a curriculum uses to represent key concepts and procedures:
  - Multiplication (e.g., number lines, tape diagrams, area models, etc.)
  - Division (e.g., number lines, tape diagrams, area models, etc.)
- Identify the language, structures, and symbolic notation a curriculum uses to emphasize the following key understandings:
  - Rewriting n x a/b as $\frac{(n \times a)}{b}$
  - Multiplying a whole number by a mixed number (e.g., Do they rewrite the mixed number as a fraction greater than one? Do they use the distributive property?)
  - Multiplying two fractions involves redefining the whole or unit
  - Understanding $a \div b$ as $\frac{a}{b}$ or $a \times \frac{1}{b}$ grounded in a sharing context
  - Rewriting equations involving division of fractions as multiplication with a missing factor equations
    - $m \div \frac{1}{n} = ?$ can be rewritten as $? \times \frac{1}{n} = m$
    - $\frac{1}{n} \div m = ?$ can be rewritten as $? \times \frac{1}{m} = n$
  - Interpreting division as finding the number of units in one group or finding the number of groups

FMD.C.2: Task Analysis

Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

Grade 4:

- We can write fraction a/b as a sum of a copies of 1/b as well as a product of a and 1/b.
- Visual representations can illustrate why n x a/b can be rewritten as $\frac{(n \times a)}{b}$.
- There are several techniques to multiply a whole number by a mixed number, such as:
  - Multiplying a whole number by a fraction greater than one (e.g., 3 x 1 ½ = 3 x 3/2)
  - Using the distributive property to multiply both addends by n (e.g., 3 x 1 ½ = (3 x 1) + (3 x ½))

What Must I Understand About My Curriculum's Approach to This Content? (FMD.C)

Grade 5:

- We can understand $a \div b$ as $\frac{a}{b}$ or $a \times \frac{1}{b}$ through real-life situations and visual representation.
- We can understand fraction a/b as a share of a whole (e.g., we can understand ⅚ as ⅙ of 5) and connect it to the understanding of a/b as a x 1/b developed through exploring fractions as sharing.
  (CCSS Writing Team, 2019, p.151)
- We can use visual representations to justify why $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.
- As with whole numbers, we can use area models to find the product of two fractions.
- We can rewrite equations involving division of fractions as multiplication with a missing factor equations.
    - $m \div \frac{1}{n} =?$ can be rewritten as $? \times \frac{1}{n} = m$
    - $\frac{1}{n} \div m =?$ can be rewritten as $? \times \frac{1}{m} = n$

When multiplying a number by a factor, the size of the factor relative to 1 determines whether the product will be greater than, equal to, or less than the size of the original number.

When dividing a number by a divisor, the size of the divisor relative to the dividend helps us predict whether the quotient will be greater than, equal to, or less than the size of the dividend.

FMD.C.3: Coherence

Determine where and how a given curriculum addresses the following standards and describe how these standards extend concepts of fraction multiplication and division:

Grade 5:

5.NBT.B.7: Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Content Focus Area:

Deconstructing Word Problems: Promoting Access to Language and Mathematical Concepts for All Learners

This content focus area takes a wider view on mathematical content than the prior modules by focusing on the role of language and contextualized problems across several domains. This approach provides two distinct opportunities: 1) Teachers can develop equitable, language-inclusive practices applicable to any mathematical content, 2) Teachers can develop a more comprehensive knowledge of structures that span students' K-8 mathematical experience. Effective math instruction integrates contextualized problems as an important part of building conceptual understanding — especially through the use of visual representations and equations to give meaning to operations. Effective math instruction also embraces the simultaneous development of language and content. This module aims to help grade 3-5 teachers understand the language complexities of all problem types and provide them with the skills needed to integrate practices that promote access to, and engagement with, language and deep mathematical understanding.

Traditionally, language support in mathematics has centered around exposure to and mastery of math vocabulary, often without plentiful opportunities for students to make sense of technical, context-specific language. While promoting access to academic vocabulary is certainly worthwhile, it represents a tiny slice of the experiences students have interacting with language in math class. Between reading, writing, listening, speaking, representing, and interacting in math, language is inextricably intertwined with content. More broadly, math content, practices, and language are interdependent (ELSF Guidelines, 2018). Just like math teachers need to understand mathematical learning progressions and support children to develop more sophisticated ways of reasoning, so, too, do they need to intentionally support children to refine mathematical language over time; the two might be considered congruent arcs for mathematics and language development in each unit. One of the most important skills teachers can develop is to help children make connections between current language, new language, and math concepts.

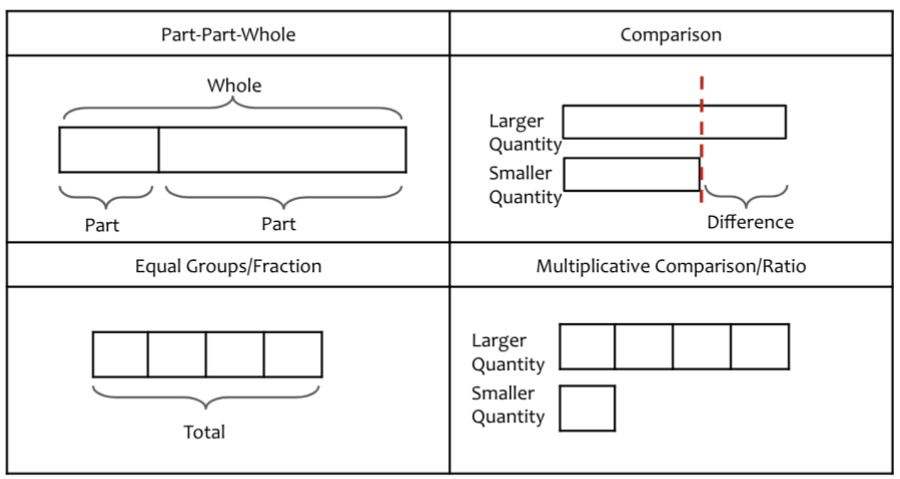
Included below are specific competencies teachers should incorporate into their pedagogy in order to capitalize on, and build, children's language resources. These content focus areas do not, however, explain the dimensions of language that help us better understand what it is that we are doing, nor do they underscore why this content focus area is so critical to supporting a variety of learners — especially multilingual learners — in terms of providing access and equitable opportunities to engage in mathematical learning. The following table — and the similar tables that can be found throughout this content focus area — provide important details about the competencies, or key instructional practices, that will help teachers better understand the purpose, function, and implications for these critical pedagogical moves.

Key Instructional Practice	Underlying Language Considerations	Implications for Access and Equity
Given the concrete–representational–abstract cycle, instruction should ensure substantial dialogue accompanies movement between representations as learners begin to abstract and generalize concepts.	Students should have a central role in producing language as they move among representations; this is a critical component of sense-making.	When teachers center discourse, participation patterns become more clear. Class culture shifts from one where learners receive explanations to one where learners produce and critique explanations.

This content focus area also aims to help teachers see the opportunities within their own curriculum to support students' language use and development through implementing the Concrete–Representational–Abstract (C-R-A) cycle, where students work with concrete objects to problem solve before using representations that can be efficiently reproduced (including tape diagrams, number lines, area models) and ultimately using abstract symbols and procedures to represent efficiently. When students use the C-R-A cycle faithfully and consistently use and support students in using representations like tape diagrams, the representations become a natural part of their mathematical thinking (Murata, 2008). Tape diagrams can be the connective tissue uniting content across grades beginning with basic additive relationships with linking cubes and weaving through comparison, multiplication and division, fraction concepts and operations, multiplicative comparison, ratio reasoning, and multi-step algebraic reasoning. Tape diagrams assist in understanding contextual problems but also provide opportunities to understand the meaning of operations, notice the inverse relationships represented by a given tape diagram, and integrate number sense to factor in the relative size of numbers in their representation as students make a plan to solve (Ng & Lee, 2009). The role that tape diagrams can play and their link from context to operations make them an essential tool to create grade level access for all students. When students consistently reason with tape diagrams, they eventually see relationships within the structure of contextual problems without needing to represent every piece of a math problem. Through their work connecting visual representations to expressions and equations, students become proficient in writing expressions and equations to efficiently represent complex problems.

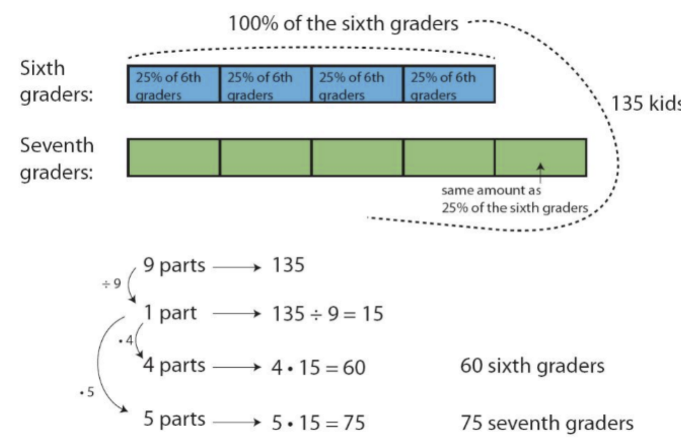
The charts below illustrate the building blocks (part–part–whole, comparison, equal groups, multiplicative comparison, and percentages — including percent increase/decrease) as well as a middle school application of those same building blocks.

Tape Diagram Structures



7th Grade Example

There are 25% more seventh graders than sixth graders in the after-school club. If there are 135 sixth and seventh graders altogether in the after-school club, how many are sixth graders and how many are seventh graders? (CCSS Writing Team, 2019, p.165)



(CCSS Writing Team, 2019, p.165)

For the reasons above and their demonstrated success in many high-performing countries, tape diagrams are now being widely adopted in American math programs. Despite the increasing popularity of tape diagrams, there is a shortage of professional learning affording teachers the opportunity to see their impact across grades and concepts.

Teachers can leverage the C-R-A cycle beyond tape diagrams. Teachers who know their curriculum's frequent contexts will be able to maximize these contexts in order to provide a solid foundation from which learners can make sense of mathematical concepts. Teachers who are familiar with hands-on-materials and visual models are better-equipped to draw on and leverage students' intuition, an under-utilized resource in many math classrooms and an excellent way to amplify language. For example, teachers can recognize how often curricula use ribbons and string in contextualized problems about fractions. Teachers who carve out space for students to engage with contexts authentically and initiate and generalize their thinking with visual models are more likely to provide abundant and meaningful opportunities for children to make sense of math concepts.

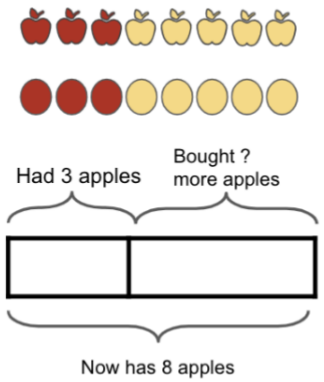
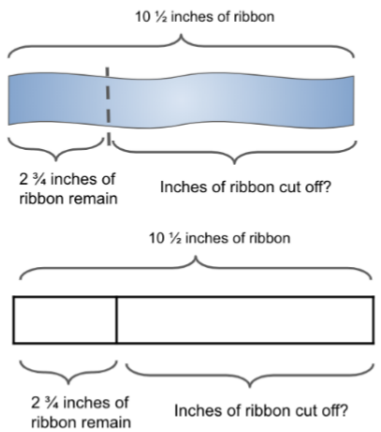
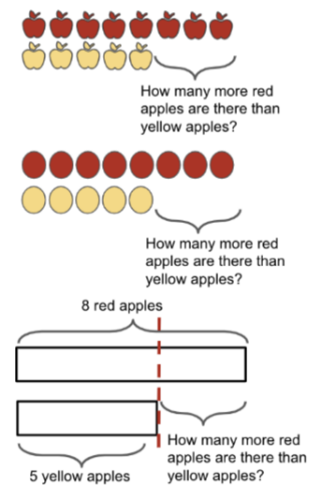
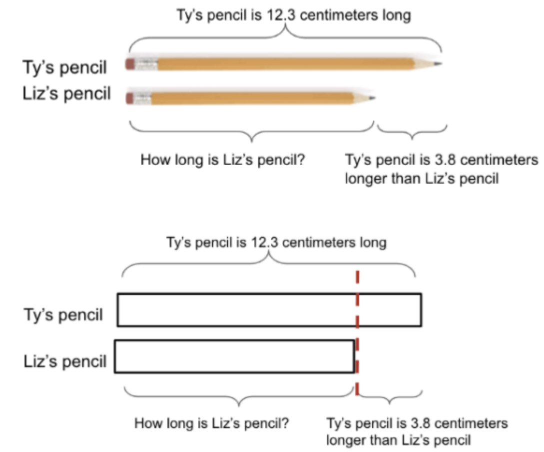
Addition and Subtraction Problem Types with Supporting Visuals (ASPT)

This competency differs from the other competencies in that it reaches back into the early elementary application standards and puts them front and center. A natural question emerges: why revisit addition and subtraction problem types that students should have internalized before their grade 3-5 math experience? In short, because the addition and subtraction application standards are not the main focus of grade 3-5 math standards, they do not often receive attention in professional learning. Nonetheless, these standards exist throughout grades 3-5, embedded in measurement contexts, fractions word problems, and multi-step word problems. Teachers who understand these basic constructs well will be able to better respond to student thinking and connect it to the newer work of equal groups, fractions, multiplicative comparison, and eventually ratio reasoning.

The Concrete–Representation–Abstract (C-R-A) cycle is re-visited in this competency to clarify how the learning progression of addition and subtraction problem types can be facilitated and expanded through visual representations. One of the goals of this grade band is to help students move fluidly among a variety of representations of addition and subtraction in order to establish the strongest possible base from which they can begin to conceptualize new math concepts. Contextualized problems and extended mathematical dialogue are two promising vehicles through which students can make sense of related representations and begin to generalize and abstract these concepts. Below are two key instructional practices that teachers ought to incorporate into their pedagogy in order to maximize sense-making opportunities.

Key Instructional Practice	Underlying Language Considerations	Implications for Access and Equity
Teachers analyze problem solving scenarios to assess for language demands and develop strategies (e.g., mediating activities; visual representations) to support understanding of the problem (ELSF Guidelines 6a, 6b).	Contexts serve as touchstones for making sense of math concepts; language — both written and oral — mediates students' understanding of the context.	Investing time and energy into supporting students' understanding of the context will help ensure that assessment of mathematical understanding is accurate, removing, to the extent possible, cases in which the language demands are obscuring mathematical understanding.
Teachers look for opportunities to draw on students' prior knowledge, culture, and experiences in order to make sense of mathematical contexts (ELSF Guideline 10B).	Tapping into familiar experiences provides students opportunities to generate language and get on solid footing to make sense of new or emergent math concepts.	Connecting everyday knowledge and experiences to new academic concepts positions students at the center of the learning experience.

The examples below of part–part–whole and comparison problem types in the contexts of K-2 mathematical content and 3-5 mathematical content may help shed light on why understanding these constructs matters. In the comparison problem featuring Ty and Liz's pencils, the problem provides one pencil length and a description of how much longer one pencil is than the other. Students are provided with the difference and must work backwards to find the smaller quantity. Thus, this problem has several language and mathematical complexities that a teacher might not fully appreciate if they see immediately that this problem can be solved by subtracting 3.8 from 12.3.

K - 2 Application of the C-R-A cycle for Addition and Subtraction	3 - 5 Application of the C-R-A cycle for Addition and Subtraction	K - 2 Application of the C-R-A cycle for Addition and Subtraction	3 - 5 Application of the C-R-A cycle for Addition and Subtraction
Part-Part-Whole		Comparison	
Joelle had 3 apples. She bought more apples. Now she has 8 apples. How many more apples did she buy?  Situation equation: 3 + ? = 8 Solution equation: 8 - 3 = ?	Myrialis has a piece of ribbon that is 10 ½ inches long. She cuts off some of her ribbon and now 2 ¾ inches remain. How many inches of ribbon did Myrialis cut off?  Situation equation: 10 ½ - ? = 2 ¾ Solution equation: 10 ½ - 2 ¾ = ?	There are 8 red apples and 5 yellow apples. How many more red apples are there than yellow apples?  Equations: 8 - 5 = ? 5 + ? = 8	Ty's pencil is 12.3 centimeters long. Ty's pencil is 3.8 centimeters longer than Liz's pencil. How long is Liz's pencil?  Equations: 12.3 - 3.8 = ? ? + 3.8 = 12.3

Prior to engaging with the individual competencies, we suggest prioritizing *ASPT.TC.3a: Select or modify existing tasks so that the C-R-A cycle can be applied (e.g., a task about a water bottle could easily be modeled using the C-R-A cycle)*. In order to thoroughly address this competency, learners should begin by looking at a series of tasks where the C-R-A cycle is explicitly written in and reflect on the opportunities it presents to students to make sense of mathematics.

Focus Standards

2.OA.A.1: Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions (e.g., by using drawings and equations with a symbol for the unknown number to represent the problem).

4.NF.B.3.d: Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators (e.g., by using visual fraction models and equations to represent the problem).

5.NF.A.2: Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators (e.g., by using visual fraction models or equations to represent the problem). Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result $2/5 + 1/2 = 3/7$, by observing that $3/7 < 1/2$.

Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category

3.MD.A.2: Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units (e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem).

3.MD.B.3: Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. For example, draw a bar graph in which each square in the bar graph might represent 5 pets.

4.MD.B.4: Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.

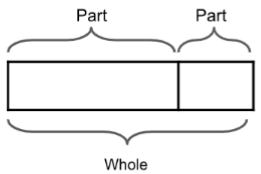
4.MD.C.7: Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems (e.g., by using an equation with a symbol for the unknown angle measure).

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)

ASPT.M.1 Understanding the Concept Progression
ASPT.M.1a: Recognize the C-R-A cycle in single-step word problem structures:
• Concrete
   o Use actual objects.
   o Use manipulatives to represent the actual objects.
• Representational
   o Draw pictures of the real objects.
   o Draw more efficient pictorial representations like circles.
   o Represent using a tape diagram.
• Abstract
   o Represent situations symbolically using expressions and equations.
ASPT.M.1b: Demonstrate the C-R-A cycle on Add To Result Unknown, Take from Result Unknown, Put Together/Take Apart Total Unknown problems.
ASPT.M.1c: Given an Add To Result Unknown, Take from Result Unknown, Put Together/Take Apart Total Unknown problem in the context of the grade 3-5 standards, draw an accurate tape diagram and write an accurate equation.
	ASPT.CC.1: Connections to Prior Learning
	ASPT.CC.1a: Understand comparison tape diagrams build logically from student experiences lining up two different objects by endpoint to compare.[2]
	
	ASPT.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Module
	Grade 3:
	ASPT.CC.2a: Identify the underlying problem type in tasks addressing picture graphs and bar graphs (e.g., in a bar graph displaying how many types of pets are at a pet store, a question might ask "how many pets there are in total?" (Put Together/Take Apart – Total Unknown) or "how many more cats are there than dogs?" (Comparison – Difference Unknown)).
	ASPT.CC.2b: Explicitly connect the bars in bar graph representations to the tapes in comparison tape diagrams to illustrate that both diagrams involve comparing the length of two quantities by lining up by end
		ASPT.TC.1: Visual Diagrams and Representations
		ASPT.TC.1a: Understand the importance of visual representations in linking the relationships between quantities in the problem with the mathematical operations needed to solve the problem and the positive effects around solidifying visual representations before equations (Jitendra et al, 1998).
		ASPT.TC.1b: Explain why tape diagrams are an effective visual model for contextualized problems, including:
		• They are length-based models and translate well to operations on the number line.
		• They link to solution strategies that can be justified on solid conceptual grounds (Beckmann, 2004, p. 42).
		• They are the visual generalization in the C-R-A cycle and can be applied across
			ASPT.SR.1 Student Misconceptions and Difficulties
			ASPT.SR.1a: Recognize the most challenging addition/subtraction-based problem types tend to be:
			• Difficulty: Students struggle with start unknown problems given that the problem leads with the unknown quantity making it challenging for students to get started.
			• Difficulty: Students struggle with additive comparison problems where the bigger or smaller quantity is unknown and the descriptor of the difference suggests the opposite operation needed. (e.g., The Johnson family ate 4 ½ pizzas for dinner on Tuesday night. They ate 1 ⅞ more pizzas on Tuesday night than Wednesday night. How many pizzas did the Johnson family eat on Wednesday night?)

[2] K.MD.A.2: Directly compare two objects with a measurable attribute in common to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.

Why Does the Math Make Sense? (ASPT.M)

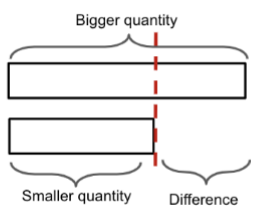


ASPT.M.1d: Demonstrate the C-R-A cycle on Add To Change Unknown or Start Unknown, Take from Change or Start Unknown, or a Put Together/Take Apart Addend Unknown problem.

ASPT.M.1e: Given an Add To Change Unknown or Start Unknown, Take from Change or Start Unknown, or a Put Together/Take Apart Addend Unknown problem in the context of the grade 3-5 standards, draw an accurate tape diagram and write an accurate equation.

ASPT.M.1f: Demonstrate the C-R-A cycle on Comparison-Difference Unknown problems.

ASPT.M.1g: Given a Comparison––Difference Unknown problem in the context of the grade 3-5 standards, draw an accurate tape diagram and write an accurate equation.

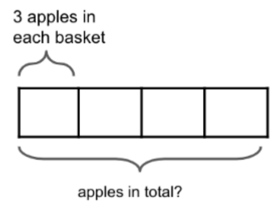


How is This Concept Connected to Other Concepts? (ASPT.CC)

points.

ASPT.CC.2c: Represent equal groups multiplication situations using tape diagrams and write an equation equating the total to the sum of the equal parts to illustrate the connection between repeated addition tape diagrams and equal groups multiplication.

e.g., $3 + 3 + 3 + 3 = 12$



Grade 4:

ASPT.CC.2d: Represent equal groups fraction multiplication situations using tape diagrams (both showing unit fractions as well as with composite units) and write an equation equating the total to the sum of the equal parts to illustrate the connection between repeated addition tape diagrams and equal groups multiplication.

e.g., $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{8}{3}$

What Must I Understand About Teaching This Content? (ASPT.TC)

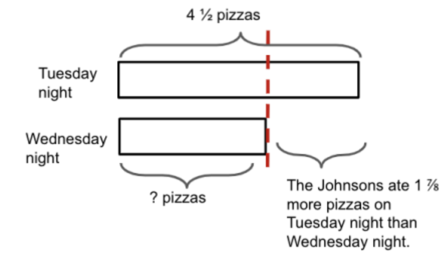
contexts (fractions, decimals, ratios, algebraic problems).

- They show the relationships between operations. (Thus, a part–whole tape diagram inherently shows that addition and subtraction can be used to reason about the relationship between the parts and the whole.)
- They can often be drawn somewhat proportionally to underscore the relative magnitude of the numbers involved in a problem.

ASPT.TC.1c: Explain the limitations of number lines as a contextualized representation of context, including:

- Students have to make decisions on which numbers to work from on the number line instead of processing the context first (which tape diagrams allow).
- Number lines do not illustrate the conceptual difference between part–part–whole structures and comparison structures.

What Must I Understand About Student Reasoning in This Content? (ASPT.SR)



ASPT.SR.1b: Recognize the multiple meanings of words frequently used in mathematical word problems and the possibility that students have not yet constructed these multiple meanings. For example:

- "Left" is a word used to convey leaving ("6 dogs left the dog park") and staying ("there were 5 dogs left at the dog park").
- "Either" and "or" are used to convey "both" at times and "one or the other but not both" at other times.

ASPT.SR.1c: Recognize problems that require students to infer mathematical units.

- Fraction problems do not always state the whole as the number 1 or as the named

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)

ASPT.M.1h: Demonstrate the C-R-A cycle on Comparison Bigger or Smaller Quantity Unknown problems.

ASPT.M.1i: Given a Comparison Bigger or Smaller Quantity Unknown problem in the context of the grade 3-5 standards, draw an accurate tape diagram and write an accurate equation.

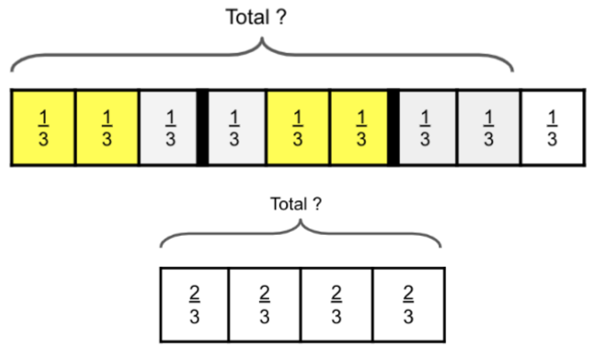
ASPT.M.1j: Given a contextualized task (featuring measurement units, whole numbers, fractions, or decimals) featuring addition or subtraction, plan a concrete representation and the tape diagram that could represent it.

ASPT.M.1k: Represent contextualized fraction addition and subtraction problems using tape diagrams (both with unit fractions and with composite units) and equations. Contextualized problems should include scenarios when the whole is not explicitly named the whole in words or with the number 1.

ASPT.M.2: Additional Math Content Knowledge for Teachers

ASPT.M.2a: Identify significant differences in problem types, including:

- Put Together/Take Apart problems do not involve an action whereas Result Unknown, Change Unknown, and Start Unknown problems do involve an action (CCSS Writing Team, 2019, pp. 17 - 19).



ASPT.CC.3 Connections to Future Learning

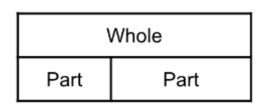
ASPT.CC.3a: Represent a one-step addition or subtraction-based algebraic situation using a tape diagram.

ASPT.TC.1d: Identify common real-life contexts for length models (which can be directly used in lessons and easily represented by tape diagrams) such as rope, ribbons, lengths of pencils, distances, etc.

ASPT.TC.1e: Identify common real-life contexts for area models such as brownies or pans of brownies, cookies, pizzas, walls to be painted, etc.

ASPT.TC.1f: Determine which visual representations are common but not as effective in representing contextualized problems, including:

- Static diagrams where the whole is double-represented (Murata, 2008, p. 385-386) and there is no distinction between part–part–whole and comparison problems



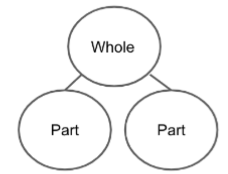
- Number bonds that show the relationships of the numbers but are not designed to show the context or action of a story nor to distinguish between

whole. Students must infer which quantity is the whole (e.g., if someone reads ⅖ of a book, we can infer that the book is 1 whole).

ASPT.SR.2 Student Supports

ASPT.SR.2a: Recognize that students with mathematics-only disabilities by definition exhibit a relative strength with language, thereby often making word problems a more accessible avenue for them to understand mathematical concepts (National Research Council, 2009, p. 111).

ASPT.SR.2b: When students struggle to make sense of a word problem structure, create a direct application of the C-R-A cycle. Examples could involve lowering the magnitude and using a context like stickers that closely resembles the counters that students could use to represent the concrete stage.

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)
• Comparison problems involve the comparison of "two distinct, disjoint sets rather than a relationship between a set and its subsets" (Carpenter et al., 2015, p. 10) indicating that comparison tape diagrams should show two quantities. • Comparison problems involve a quantity (the difference) which "is not present in the situation physically" (CCSS Writing Team, 2019, p. 21). **ASPT.M.2b**: Given a grade 3-5 addition or subtraction-based task, indicate what type of problem it represents.[1]		part–part–whole and comparison problem types  **ASPT.TC.2: Symbolic Diagrams and Representations** **ASPT.TC.2a**: Craft solution equations and situation equations (if different from the solution equation). **ASPT.TC.2b**: Explain the limitations of equations as a representation, including: • Equations may mask a misunderstanding because they don't require the attention to relationships that tape diagrams do (e.g., an equation might not provide enough information to know whether the student is understanding a problem as part–part–whole or comparison — if students don't develop the tools to visually	

[1] Students do not need to know or use the names of problem types (e.g., Add To-Start Unknown).

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)
		represent relationships, they may struggle with multi-step representation later on). **ASPT.TC.2c**: Explain the strength of equations as a representation: ● Equations can represent information more concisely and efficiently than tape diagrams. ● Equations provide a clear connection to the operations needed for solving. **ASPT.TC.3: Recommended Instructional Competencies** **ASPT.TC.3a**: Select or modify existing tasks so that the C-R-A cycle can be applied (e.g., a task about a water bottle could easily be modeled using the C-R-A cycle). **ASPT.TC.3b**: Implement think-pair-shares where students share their approaches to contextualized problems. **ASPT.TC.3c**: Select and sequence student work that could highlight an important mathematical connection (such as the relationship between addition and subtraction, the relative size of the tapes used given the	

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)
		relative magnitude of the numbers, etc.). **ASPT.TC.3d**: Pose purposeful questions that invite students to make important mathematical connections between visual and abstract representations. **ASPT.TC.3e**: Provide a consistent approach to contextualized problems such as Read–Draw–Write, Three Reads, Visualize–Represent–Solve, etc. **ASPT.TC.3f**: Recognize the language demands of a given task: • Reading • Writing • Listening • Speaking • Representing • Interacting **ASPT.TC.3g**: Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development. **ASPT.TC.3h**: Adapt common real-life contexts for addition and subtraction	

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)
		word problems to connect to student prior knowledge and experience. **ASPT.TC.4: Recommended Tasks for Promoting Understanding** **ASPT.TC.4a**: Given a tape diagram with labeled quantities, write a story problem that could match the tape diagram. **ASPT.TC.4b**: Given an addition or subtraction contextualized problem, draw a visual representation and write an equation that could be used to solve for the unknown. **ASPT.TC.4c**: Given a contextual math problem without the question stem, generate a list of possible questions. Share the possible questions with a thought partner.[3] **ASPT.TC.4d**: Given a contextual problem, describe the context to a partner, describe the meaning of the quantities in the problem, and generate possible solution methods.[4]	

[3] This brief description alludes to the Mathematical Language Routine: Co-Craft Questions. The full routine can be found here: https://achievethecore.org/page/3164/mathematical-routines.

[4] This brief description alludes to the Mathematical Language Routine: Three Reads. The full routine can be found here: https://achievethecore.org/page/3164/mathematical-routines. Note that there are many variations of Three Reads that could be considered.

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)
		ASPT.TC.5: Common Instructional Misconceptions **ASPT.TC.5a**: Misconception: We can use "key words" such as "more" means "add" to help students find the solution to word problems. Correction: This approach often does not work and it detracts from students learning to solve word problems through careful reasoning (e.g., Sarah has 4 stickers. How many more stickers does she need to fill a page that can hold 9 stickers?). **ASPT.TC.5b**: Misconception: Many non-High Quality Instruction Materials do not address the breadth of the word problem standards. (They heavily favor result unknown and comparison–difference unknown scenarios and thereby limit student exposure to grade level content.) Correction: Ensure materials reflect a balance of word problems appropriate to the grade level and sequence of the curriculum. • **ASPT.TC.5c**: Misconception: Producing correct answers to contextualized problems can	

Why Does the Math Make Sense? (ASPT.M)	How is This Concept Connected to Other Concepts? (ASPT.CC)	What Must I Understand About Teaching This Content? (ASPT.TC)	What Must I Understand About Student Reasoning in This Content? (ASPT.SR)
		be proceduralized. Correction: Students must learn to reason through a range of problems and check their work for reasonableness. Additional time should be provided to discuss representation strategies for contextualized problems.	

What Must I Understand About My Curriculum's Approach to This Content? (ASPT.C)

ASPT.C.1: Curriculum Design

- **ASPT.C.1a**: Identify the vocabulary and definitions a curriculum uses for key concepts and procedures:
  - Unknown
  - Problem Types (Does the curriculum distinguish between comparison and part–part–whole in student-facing materials?)
  - Tape Diagrams (also called Bar Models or Strip Diagrams)
- **ASPT.C.1b**: Identify real-life contexts used consistently with key concepts and procedures:
  - Addition and subtraction of whole numbers
  - Addition and subtraction of decimal numbers (e.g., metric weights, money, etc.)
  - Addition and subtraction of fractions (e.g., pizzas, length of pencils, races, etc.)
- **ASPT.C.1c**: Identify which visual representations a curriculum uses to represent addition and subtraction word problems (e.g., tape diagrams, number bonds, other schematic representations, etc.).
- **ASPT.C.1d**: Identify the language, structures, and symbolic notation a curriculum uses to emphasize the following key understandings:
  - Solving application problems requires a clear, consistent reasoning process. (E.g., Is there a prescribed approach such as "read, draw, write," "visualize, represent, solve," etc.?)
  - Does the curriculum differentiate between part–part–whole and comparison problems?

ASPT.C.2: Task Analysis

ASPT.C.2a: Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

- Tape diagrams can help us represent addition as increasing a quantity or combining of two groups.
- Tape diagrams can help us represent subtraction as decreasing a quantity or finding the difference.
- We can visually represent addition or subtraction problems with tape diagrams to understand the context and make a plan for solving.
- When thinking about equal groups, we can use repeated addition or multiplication.

Grades 4/5:

- Sometimes the whole in a fractions problem will not be directly referred to as the whole or by the number 1. Students must use context to infer which quantity is the whole.

ASPT.C.2b: Select tasks from a curriculum that pose questions from these problem structures and plan to have students share their representations to surface and discuss any misconceptions:

- Comparison where the descriptor of the difference suggests the opposite operations (e.g., when the larger quantity is known and the difference is described using the word "more," students often jump to addition)
- Start unknown problems

ASPT.C.3: Coherence

ASPT.C.3a: Analyze problems addressing the following standards to understand which addition and subtraction problem types are featured:

- 3.MD.A.2: Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units (e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem).
- 3.MD.B.3: Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. For example, draw a bar graph in which each square in the bar graph might represent 5 pets.
- 4.MD.B.4: Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.

  4.MD.C.7: Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems (e.g., by using an equation with a symbol for the unknown angle measure).

Multiplication and Division Problem Types with Supporting Visuals (MDPT)

In grade 3, students begin work with multiplication and division grounded in contextualized problems. In most curricula, the C-R-A cycle will be integrated clearly or will closely enough align to allow small modifications to enact the C-R-A cycle. In order to optimize students' opportunities to access and engage with these emergent mathematical ideas and new forms of language, teachers must work to bring these concepts to life and facilitate a learning progression that is moving toward generalization and abstraction.

Given that new content begets new language and vice versa, students should be given ample opportunity to engage through reading, writing, listening, speaking, and representing. Moreover, where meaning is initially made through existing language experiences and resources, teachers should be prepared to introduce and layer on new, formal language (i.e., make explicit connections between existing and new language) to help students make clearer meaning of mathematical ideas. For example, students may work on contextualized equal groups multiplication problems for several days before they are introduced to the formal language (e.g., multiplication, factors) and formal notation (a x b = c). The same holds true in grade 4 when students begin work with multiplicative comparison. As emphasized elsewhere, classroom experiences that are language-rich and engage students through multiple modalities support dynamic, mathematical meaning making. The following table delineates the language considerations and implications of this instructional practice.

Key Instructional Practice	Underlying Language Considerations	Implications for Access and Equity
Teachers regularly design math learning experiences where reading, writing, speaking, and listening are embedded (ELSF Guideline 4D).	Providing opportunities to produce and process language accentuates the math learning experience by incorporating both meta-linguistic awareness and meta-cognition.	Incorporating multi-modalities expands what constitutes math teaching and learning to include more than just skills-based instruction of memorization of algorithms; this increases the likelihood of greater participation and deeper understanding.

Though proficiency with writing expressions and equations to represent situations is an intended outcome of upper elementary mathematics, it should not be rushed before students have a solid conceptual understanding. Grade 3 students reasoning about division will gain more from representing concretely and visually with attention to which quantity is the number of groups and which is the number of units in each group than by simply recording a division equation that doesn't require the depth of understanding.

Focus Standards
3.OA.A.3: Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities (e.g., by using drawings and equations with a symbol for the unknown number to represent the problem). **4.OA.A.2:** Multiply or divide to solve word problems involving multiplicative comparison (e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison). **4.NF.B.4.c:** Solve word problems involving multiplication of a fraction by a whole number (e.g., by using visual fraction models and equations to represent the problem). For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie? **5.NF.B.3:** Interpret a fraction as division of the numerator by the denominator (a/b = a ÷ b). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers (e.g., by using visual fraction models or equations to represent the problem). For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie? **5.NF.B.6:** Solve real world problems involving multiplication of fractions and mixed numbers (e.g., by using visual fraction models or equations to represent the problem). **5.NF.B.7.c:** Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions (e.g., by using visual fraction models and equations to represent the problem). For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?
Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category
3.NF.A.3b: Recognize and generate simple equivalent fractions (e.g., 1/2 = 2/4, 4/6 = ⅔). Explain why the fractions are equivalent (e.g., by using a visual fraction model). **3.NF.A.3d:** Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions (e.g., by using a visual fraction model). (All fractions standards involving tape diagrams or visual models are not referenced here.) **3.MD.A.2:** Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units (e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem). **5.MD.B.2:** Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.

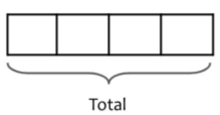
Why Does the Math Make Sense? (MDPT.M)	How is This Concept Connected to Other Concepts? (MDPT.CC)	What Must I Understand About Teaching This Content? (MDPT.TC)	What Must I Understand About Student Reasoning in This Content? (MDPT.SR)

MDPT.M.1: Understanding the Concept Progression

Grade 3:

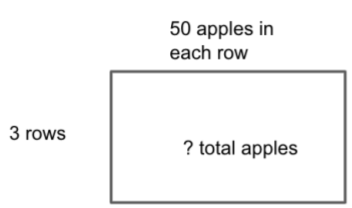
MDPT.M.1a: Demonstrate the C-R-A cycle on Equal Groups–Unknown Product problems.

MDPT.M.1b: Given an Equal Groups– Unknown Product problem, draw an accurate tape diagram and write an accurate equation.



MDPT.M.1c: Demonstrate the C-R-A cycle on Area/Array–Unknown Product or Area/Array–Unknown Factor problems.

MDPT.M.1d: Given an Area/Array–Unknown Product or Area/Array–Unknown Factor problem in the context of the grade 3-5 standards, draw an accurate tape diagram and write an accurate equation.

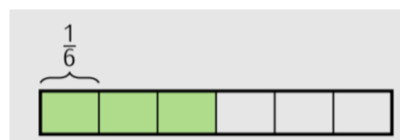


MDPT.CC.1: Connections to Prior Learning

**See prior sub-categories on Language in Mathematics: Promoting Access for All Learners.*

MDPT.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Module

MDPT.CC.2a: Use tape diagrams to represent fraction a/b. Explain how the tape diagram shows the repeated addition of 1/b a times as well as the relationship a x 1/b.



(CCSS Writing Team, 2019, p.142)

MDPT.CC.3: Connections to Future Learning

MDPT.CC.3a: Show that multiplicative comparison tape diagrams can also be used to show reasoning with ratios that students will learn in grade 6.

MDPT.TC.1: Visual Diagrams and Representations

MDPT.TC.1a: Explain why tape diagrams are an effective visual model for contextualized multiplication and division problems, including:

- They are length-based models and translate well to operations on the number line.
- They link to solution strategies, including repeated addition and repeated subtraction (which are often entry points for students with weak multiplicative reasoning).
- They visually demonstrate the difference between measurement division and partitive division.
- They naturally encompass fractions as an equal group model and allow students to see the multiplicative relationship.

MDPT.TC.1b: Explain the difference between area models and tape diagrams, including:

- Tape diagrams illustrate equal groups problems well.

MDPT.SR.1: Student Misconceptions and Difficulties

MDPT.SR.1a: Recognize common difficulties relating to division such as:

- Difficulty: Students struggle to decide whether the problem is describing a context with the number of units in 1 group unknown or the number of groups unknown.

MDPT.SR.1b: Recognize the most challenging multiplication/division-based problem types tend to be:

- Difficulty: Students struggle to represent multiplicative comparison problems where the bigger or smaller quantity is unknown and the descriptor of the multiplier suggests the opposite operation needed. (E.g., Jada has 120 stickers. Jada has 8 times as many stickers as Ashish. How many stickers does Ashish have?)

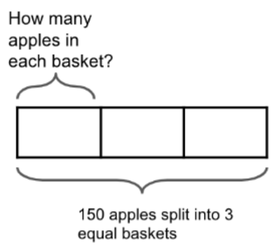
MDPT.SR.1c: Difficulty: Students

Why Does the Math Make Sense? (MDPT.M)	How is This Concept Connected to Other Concepts? (MDPT.CC)	What Must I Understand About Teaching This Content? (MDPT.TC)	What Must I Understand About Student Reasoning in This Content? (MDPT.SR)

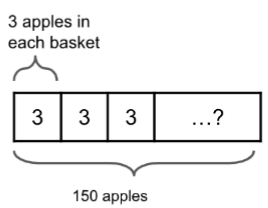
MDPT.M.1e: Demonstrate the C-R-A cycle on Equal Groups–Number of Groups Unknown and Group Size Unknown problems.

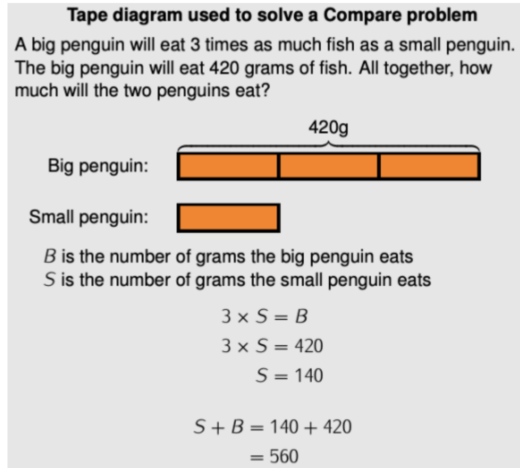
MDPT.M.1f: Given an Equal Groups–Number of Groups Unknown or Group Size Unknown problem, draw an accurate tape diagram and write an accurate equation.

- Equal Groups–Group Size Unknown



- Equal groups–Number of Groups Unknown[1]





(CCSS Writing Team, 2019, p. 38)

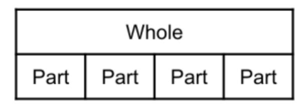
E.g., the above tape diagram also shows that the ratio of fish consumption of big penguins to small penguins is 3:1. This tape diagram could be used to figure out how much food each penguin ate if 440 grams of fish were eaten.[3]

Use a tape diagram to represent an algebraic situation in the form px = q.[4]

- Area models illustrate problems grounded in rows and columns and area contexts well.

MDPT.TC.1c: Explain the limitations of static diagrams, including:

- The whole is double represented (Murata, 2008, p. 385-386).
- There is no distinction between equal groups and multiplicative comparison.
- This structure does not easily apply to fractions given that it appears to double the size of the whole.



MDPT.TC.2: Symbolic Diagrams and Representations

MDPT.TC.2a: Explain the limitations of equations as a representation, including:

- Equations may mask a misunderstanding because they

struggle to recognize that two quantities can be compared in two ways (“how much more” and “how many times more?”) and sometimes struggle to differentiate between the two.

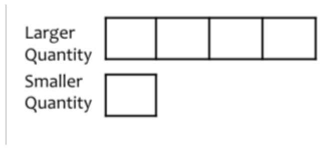
MDPT.SR.2 Student Supports

MDPT.SR.2a: When students struggle to make sense of a word problem structure, create a direct application of the C-R-A cycle. Examples involving equal groups division could include asking a question such as, “Bryan has 20 counters and 5 cups. If he wants to use all of his counters to put the same number of counters in each cup, how many counters will be in each cup?” and letting students represent concretely with counters and cups before connecting to the tape diagram and equation.

[1] The tape diagram used features an ellipse to indicate that the number of groups keeps going. Tape diagrams can also show Equal Groups–Number of Groups Unknown by showing all the groups if the number of groups is small enough to be shown.

[3] 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems (e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations).

[4] 6.EE.B.7: Solve real-world and mathematical problems by writing and solving equations of the form x + p = q and px = q for cases in which p, q, and x are all nonnegative rational numbers.

Why Does the Math Make Sense? (MDPT.M)	How is This Concept Connected to Other Concepts? (MDPT.CC)	What Must I Understand About Teaching This Content? (MDPT.TC)	What Must I Understand About Student Reasoning in This Content? (MDPT.SR)
Grade 4: **MDPT.M.1g**: Demonstrate the C-R-A cycle on Multiplicative Comparison problems. **MDPT.M.1h**: Given a Multiplicative Comparison problem, draw an accurate tape diagram and write an accurate equation.  **MDPT.M.2: Additional Math Content Knowledge for Teachers** **MDPT.M.2a**: Recognize that the Equal Groups tape diagram is also the foundation for fractions. **MDPT.M.2b**: Given a multiplication or division-based task, indicate what type of problem it represents.[2] **MDPT.M.2c**: Given a problem type, identify whether an equal groups tape diagram or area/array model is more appropriate.		don't require the attention to relationships that tape diagrams do. (E.g., an equation might not provide enough information to know whether the student is understanding whether the number of groups is unknown or the number of units in 1 group is unknown.) **MDPT.TC.2b**: Explain the strength of equations as a representation: • Equations can represent information more concisely and efficiently than tape diagrams. • Equations provide a clear connection to the operations needed for solving. **MDPT.TC.3: Recommended Instructional Competencies** **MDPT.TC.3a**: Recognize the language demands of a given task: • Reading • Writing • Listening • Speaking	Students then move to more abstract contexts.

[2] Students do not need to know or use the names of problem types (e.g., Multiplicative Comparison–Smaller Unknown).

Why Does the Math Make Sense? (MDPT.M)	How is This Concept Connected to Other Concepts? (MDPT.CC)	What Must I Understand About Teaching This Content? (MDPT.TC)	What Must I Understand About Student Reasoning in This Content? (MDPT.SR)
		- Representing - Interacting **MDPT.TC.3b**: Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical Language Routines to enact existing tasks while promoting language access and development. **MDPT.TC.3c**: Select or modify existing tasks so that the C-R-A cycle can be applied. (E.g., a task about a water bottle could easily be modeled using the C-R-A cycle.) **MDPT.TC.3d**: Implement think-pair-shares where students share their approaches to contextualized problems. **MDPT.TC.3e**: Adapt common real-life contexts for multiplication and division word problems to connect to student prior knowledge and experience. **MDPT.TC.4: Recommended Tasks for Promoting Understanding** **MDPT.TC.4a**: Represent Equal Groups and Area/Array word problems with real-life objects, linking cubes, foam tiles, tape diagrams, arrays, and equations.	

Why Does the Math Make Sense? (MDPT.M)	How is This Concept Connected to Other Concepts? (MDPT.CC)	What Must I Understand About Teaching This Content? (MDPT.TC)	What Must I Understand About Student Reasoning in This Content? (MDPT.SR)
		MDPT.TC.4b: Represent the Multiplicative Comparison word problems with real-life objects, linking cubes, tape diagrams, and equations. **MDPT.TC.4c**: Given a multiplication/division tape diagram with labeled quantities, write a story problem that could match the tape diagram. **MDPT.TC.4d**: Given a multiplication or division contextualized problem, draw a visual representation and write an equation that could be used to solve for the unknown. **MDPT.TC.4e**: Given a contextual math problem without the question stem, generate a list of possible questions. Share the possible questions with a thought partner.[5] **MDPT.TC.4f**: Given a contextual problem, describe the context to a partner, describe the meaning of the quantities in the problem and generate possible solution methods.[6]	

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		MDPT.TC.5: Common Instructional Misconceptions **MDPT.TC.5a**: Misconception: We can use “key words” such as “more” means “add” to help students find the solution to word problems. Correction: This approach often does not work and it detracts from students learning to solve word problems through careful reasoning. (E.g., Sarah has 40 stickers. Each of the pages in her sticker album holds 8 stickers. How many pages of Sarah’s sticker album would be needed to hold all of Sarah’s stickers?) **MDPT.TC.5b**: Misconception: Many non-High Quality Instructional Materials do not address the breadth of the word problem standards. (They heavily favor equal groups multiplication and division and may limit multiplicative comparison, thereby limiting student exposure to grade level content.) Correction: Supplement word problem sets with a more representative mix of word problem types. **MDPT.TC.5d**: Misconception: Many non-High Quality Instructional Materials often include rate language prematurely. (E.g., Julia can run 6 miles per day. How many	

Why Does the Math Make Sense? (MDPT.M)	How is This Concept Connected to Other Concepts? (MDPT.CC)	What Must I Understand About Teaching This Content? (MDPT.TC)	What Must I Understand About Student Reasoning in This Content? (MDPT.SR)
		miles can she run in 4 days?) Rate language with continuous quantities is more appropriate for grade 6. Correction: Adjust the problems so that there is no rate language. Example: Every day Julia runs 6 miles. How many miles would Julia run in 4 days? **MDPT.TC.5d**: Misconception: Producing correct answers to contextualized problems can be proceduralized. Correction: Students must learn to reason through a range of problems and check their work for reasonableness. Teachers should provide additional time to discuss representation strategies for contextualized problems.	

What Must I Understand About My Curriculum's Approach to This Content? (MDPT.C)

MDPT.C.1: Curriculum Design

- **MDPT.C.1a:** Identify the vocabulary and definitions a curriculum uses for these concepts and procedures:
  - Partitive division (e.g., equal sharing, etc.)
  - Measurement division (e.g., quotative division, etc.)
  - Tape Diagrams (also called Bar Models or Strip Diagrams)
- **MDPT.C.1b**: Identify real-life contexts used consistently with key concepts and procedures:
  - Equal groups multiplication and division of whole numbers
  - Area/array multiplication and division of numbers, including fractions
- **MDPT.C.1c**: Identify which visual representations a curriculum uses to represent addition and subtraction word problems (e.g., tape diagrams, number bonds, other schematic representations, etc.).
- **MDPT.C.1d:** Identify the language, structures, and symbolic notation a curriculum uses to emphasize the following key understandings:
  - Solving application problems requires a clear, consistent reasoning process. (E.g., Is there a prescribed approach such as "read, draw, write," "visualize, represent, solve," etc.?)

MDPT.C.2: Task Analysis

MDPT.C.2a: Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

Grade 3:

- Tape diagrams can help us represent equal groups multiplication.
- Tape diagrams can help us differentiate the number of groups and the number of units in each group when representing division.
- Tape diagrams can help us identify operations we could use to solve a problem.

Grade 4:

- We can compare quantities in two ways: "how much more" and "how many times more?"

MDPT.C.3: Coherence

MDPT.C.3a: Analyze problems addressing the following standards to understand which multiplication and division problem types are featured.

- 3.MD.A.2: Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units (e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem).
- 5.MD.B.2: Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.

Multi-Step Problems (MSPT)

Multi-step problems tie together all of the single-step problem structures. The variation in multi-step problems underscores that representing and solving cannot be proceduralized, or routinized, to the point of automaticity the way computing with algorithms can. Simply put, students must know how to reason through and account for all necessary information in a given context and also step back to assess whether their pathways and answers are reasonable. Given the complexity of these problems, the work with reasoning and representing must be cultivated throughout a student's elementary math experience. Teachers can support these experiences by readily providing opportunities to discuss and evaluate solution methods. Again, intentionally curating language experiences directly and profoundly supports mathematical meaning-making.

Focus Standards
3.OA.D.8: Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. **4.OA.A.3:** Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.
Other Relevant 3rd - 5th Grade Level Standards to This Sub-Category
4.MD.A.2: Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale. **5.MD.A.1**: Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Why Does the Math Make Sense? (MSPT.M)	How is This Concept Connected to Other Concepts? (MSPT.CC)	What Must I Understand About Teaching This Content? (MSPT.TC)	What Must I Understand About Student Reasoning in This Content? (MSPT.SR)
MSPT.M.1: Understanding the Concept Progression **MSPT.M.1a**: Recognize a multi-step problem as a series of single-step problems and recognize which types of single-step problems are present in a multi-step problem. **MSPT.M.1b**: Accurately represent multi-step problems using tape diagrams (for applicable steps) and write both a single expression or a series of equations to represent the multiple steps. **MSPT.M.2: Additional Math Content Knowledge for Teachers** **MSPT.M.2a**: Identify whether a problem requires an embedded step or can be completed in a linear sequence.	**MSPT.CC.1: Connections to Prior Learning** **See prior sub-categories on Language in Mathematics: Promoting Access for All Learners.* **MSPT.CC.1a**: Represent and solve two-step problems involving only addition and subtraction.[1] **MSPT.CC.2: Connections to Other Relevant 3rd - 5th Grade Level Standards to This Module** **MSPT.CC.3: Connections to Future Learning** **MSPT.CC.3a**: Represent two-step problems using tape diagrams and equations involving a variable.	**MSPT.TC.1: Symbolic Diagrams and Representations** **MSPT.TC.1a**: Explain how grouping symbols can be useful for including embedded steps of reasoning within a larger arc of reasoning. **MSPT.TC.1b**: Explain the necessity of variables to account for different pieces of unknown information in multi-step problems. **MSPT.TC.2: Recommended Instructional Competencies** **MSPT.TC.2a**: Recognize the language demands of a given task: • Reading • Writing • Listening • Speaking • Representing • Interacting **MSPT.TC.2b**: Plan to enact tasks addressing multiple language modalities. Teachers can apply the Mathematical	**MSPT.SR.1: Student Misconceptions** **MSPT.SR.1a**: Misconception: Students record multiple steps of reasoning in a running equation such as 4 x 5 = 20 + 2 = 22. Correction: Instead, students should record reasoning in separate equations: 4 x 5 = 20. 20 + 2 = 22 or an equation such as (4 x 5) + 2 = 20. **MSPT.SR.2: Student Supports** **MSPT.SR.2a**: Provide guidance on multi-step problems that benefit students. **MSPT.SR.2b**: Underline the last line of problems and write an answer sentence with a blank unknown. **MSPT.SR.2c**: Provide a structure such as Three Reads that will help students process the problem before solving.

[1] 2.OA.A.1: Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions (e.g., by using drawings and equations with a symbol for the unknown number to represent the problem).

Why Does the Math Make Sense? (MSPT.M)	How is This Concept Connected to Other Concepts? (MSPT.CC)	What Must I Understand About Teaching This Content? (MSPT.TC)	What Must I Understand About Student Reasoning in This Content? (MSPT.SR)
		Language Routines to enact existing tasks while promoting language access and development. **MSPT.TC.2c:** Select or modify existing tasks so that the C-R-A cycle can be applied. (E.g., a task about a water bottle could easily be modeled using the C-R-A cycle.) **MSPT.TC.2d:** Implement think-pair-shares where students share their approaches to contextualized problems. **MSPT.TC.2e:** Anticipate multiple correct solution pathways and several possible misconceptions. **MSPT.TC.2f:** Adapt common real-life contexts for multi-step word problems to connect to student prior knowledge and experience. **MSPT.TC.3: Recommended Tasks for Promoting Understanding** **MSPT.TC.3a:** Represent multi-step word problems with visual representations and expressions or equations. **MSPT.TC.3b:** Given a tape diagram with labeled quantities or a multi-step equation or expression, write a story problem that could match the	

Why Does the Math Make Sense? (MSPT.M)	How is This Concept Connected to Other Concepts? (MSPT.CC)	What Must I Understand About Teaching This Content? (MSPT.TC)	What Must I Understand About Student Reasoning in This Content? (MSPT.SR)
		representation. **MSPT.TC.3c:** Given a contextual math problem without the question stem, generate a list of possible questions. Share the possible questions with a thought partner.[2] **MSPT.TC.3d:** Given a word problem, describe the context to a partner, describe the meaning of the quantities in the problem, and generate possible solution methods.[3] **MSPT.TC.4: Common Instructional Misconceptions** **MSPT.TC.4a:** Misconception: Students need to recognize how many steps there are in a problem and separate their work to show each separate step. Correction: Multi-step problems can at times be clearly represented with one visual representation and other times lend themselves to multiple visual representations. Formulaic approaches such as dividing a page into two sections for two steps should be discouraged.	

[2] This brief description alludes to the Mathematical Language Routine: Co-Craft Questions. The full routine can be found here: https://achievethecore.org/page/3164/mathematical-routines.

[3] This brief description alludes to the Mathematical Language Routine: Three Reads. The full routine can be found here: https://achievethecore.org/page/3164/mathematical-routines. Note that there are many variations of Three Reads that could be considered.

Why Does the Math Make Sense? (MSPT.M)	How is This Concept Connected to Other Concepts? (MSPT.CC)	What Must I Understand About Teaching This Content? (MSPT.TC)	What Must I Understand About Student Reasoning in This Content? (MSPT.SR)
		MSPT.TC.4b: Misconception: Producing correct answers to contextualized problems can be proceduralized. Correction: Students must learn to reason through a range of problems and check their work for reasonableness. Teachers should provide additional time to discuss representation strategies for contextualized problems. **MSPT.TC.4c**: Misconception: Reasoning with multiple calculations can be recorded using a running equation such as: 4 x 5 = 20 + 2 = 22. Correction: Reasoning should be recorded in separate equations: 4 x 5 = 20. 20 + 2 = 22 or using an equal sign only when expressions are actually equivalent (e.g., (4 x 5) + 2 = 22.)	

What Must I Understand About My Curriculum's Approach to This Content? (MSPT.C)

MSPT.C.1: Curriculum Design

- **MSPT.C.1a**: Identify the vocabulary and definitions a curriculum uses for these key concepts and procedures:
    - Variable
    - Grouping symbols
- **MSPT.C.1b**: Identify which visual representations a curriculum uses to explain addition and subtraction of fractions and determine whether they are length-based or area-based (e.g., number lines, area models, etc.).
- **MSPT.C.1c**: Identify the language, symbolic notation, and strategies a curriculum uses to emphasize the following key understandings:
    - Solving application problems requires a clear, consistent reasoning process. (E.g., is there a prescribed approach such as "read, draw, write," "visualize, represent, solve," etc.?)

MSPT.C.2: Task Analysis

MSPT.C.2a: Select tasks from a curriculum that address the following key understandings and plan a clear, mathematically-accurate concept synthesis in words and/or an anchor chart that makes the mathematics explicit.

- We can understand multi-step problems as a series of one-step problems.
- Tape diagrams can help us visualize relationships between quantities.
- We can use grouping symbols to convey embedded problem solving steps within a longer expression or equation.
- Problem solving involves stepping back and assessing the reasonableness of pathways and solutions.

MSPT.C.3: Coherence

MSPT.C.3a: Represent problems addressing the following standards using tape diagrams (only if applicable) and expressions or equations.

- 4.MD.A.2: Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.
- 5.MD.A.1: Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Curriculum Understanding: Overarching Competencies (CU)

Curriculum understanding is deeply integrated with an understanding of the content knowledge and pedagogical content knowledge being taught.
Throughout the mathematics competencies, there are detailed competencies that illustrate how the unique mathematics content should be learned and prepared for implementation in partnership with a high-quality curriculum. That said, there are some common and general upfront knowledge, skills, and mindsets necessary for success with a high-quality curriculum. Below are the general Curriculum Understanding: Overarching Competencies needed for all teachers using a high-quality curriculum.

Curriculum Understanding: Prepare to Access Curriculum (CU.AC)

CU.AC.1: Access the curriculum through understanding of all of the required elements of the curriculum and the ability to find and organize all of the elements in their classroom.

CU.AC.2: Identify the additional necessary materials (e.g. manipulatives, novels), describe when they are required within the curriculum, and organize them based on their necessary use in the classroom.

Curriculum Understanding: Internalize and Appropriately Use Curriculum Design (CU.CD)

CU.CD.1: Identify the parts of the curriculum (e.g., lesson, section, unit, course) and explain how they fit together.

CU.CD.2: Explain the organization of the curriculum at each level, including when and how to use teacher-facing materials and student-facing materials.

CU.CD.3: Describe the student expectations of the curriculum through student work samples and evaluate how personal beliefs and expectations for students align with the curriculum expectations and adjust as needed.

CU.CD.4: Explain how the curriculum assesses student learning, including identifying and describing the information gathered, how that information is gathered and how often.

CU.CD.5: Identify how the curriculum is aligned to standards at each level (i.e., day/lesson, weeks/units, year/course) and can explain how the curriculum approaches standards alignment and assessment (i.e., Each lesson is aligned to a group of standards which are revisited across multiple days and then assessed at the end of the section in a unit).

CU.CD.6: Describe the content-specific curricular approaches and routines.
- ELA specific: use of texts, approach to knowledge building, vocabulary development, writing progression
- Math specific: content routines, mathematical language routines

CU.CD.7: Describe the approach the curriculum uses and identify the content and the classroom structures needed to build students' skill, particularly when students have gaps in their learning and need additional instruction and time with content and skill to be able to meet the expectations of the curriculum.

CU.CD.8: Identify the pedagogical strategies, approaches, and/or frameworks (e.g., personalized or blended learning, incorporation of 1 to 1 technology) the curriculum uses and determine the preparation (e.g., classroom setup, access to technology) and/or learning they need to employ those strategies, approaches, and/or frameworks.

CU.CD.9: Describe how the curriculum incorporates culturally relevant practices and addresses social emotional learning.

CU.CD.10: Identify the elements of the curriculum most likely to be omitted and the causes of that in order to maintain all critical elements of the curriculum.

Curriculum Understanding: Prepare to Effectively Implement Curriculum (CU.CI)

CU.CI.1: Identify the necessary timing of the curriculum and determine, based on a schedule and calendar, how to pace delivery of the curriculum, referring to any school or district guidance on pacing.

CU.CI.2: Identify the expectations of the curriculum for using information gained from the assessments, determine how to ensure students are gaining the knowledge and skills necessary to meet the expectations of the curriculum, and explain the instructional options for helping all students, particularly those who learn in a different way and at a different pace than their peers.

CU.CI.3: Describe the purpose of monitoring students' learning across the curriculum, specifically describing its importance in using that information to determine how to prepare each student to access the lesson with their peers. Identify the specific tasks and expectations necessary to prove a student is mastering the content. Describe potential gaps that may occur in task completion and what that would indicate about student support necessary.

CU.CI.4: Understand a school's/district's grading policies and identify which tasks in the curriculum will meet the necessary requirements. Leverage tasks effectively to fulfill grading requirements.

CU.Ci.5: Identify what parents want to know about their child's curriculum (e.g. homework expectations, where to find answer to support them, how to monitor success). Identify within the curriculum the specific elements that will help a parent answer their questions about the curriculum and support their child. Communicate this to a parent in a clear and easy to access way.

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Appendix

Grade 3 Standards	Content Focus Area	Module	Priority
3.OA.A.1: Represent and solve problems involving multiplication and division. Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7.	Language in Mathematics	Multiplication and Division Problem Types	
3.OA.A.2: Represent and solve problems involving multiplication and division. Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.	Language in Mathematics	Multiplication and Division Problem Types	
3.OA.A.3: Represent and solve problems involving multiplication and division. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities (e.g., by using drawings and equations with a symbol for the unknown number to represent the problem).	Language in Mathematics	Multiplication and Division Problem Types	
3.OA.A.4: Represent and solve problems involving multiplication and division. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations: 8 × ? = 48, 5 = __ ÷ 3, 6 × 6 = ?.	Multiplication and Division	Properties	
3.OA.B.5: Understand properties of multiplication and the relationship between multiplication and division. Apply properties of operations as strategies to multiply and divide. Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known (Commutative property of multiplication). 3 × 5 × 2 can be found by 3 × 5 = 15 then 15 × 2 = 30, or by 5 × 2 = 10 then 3 × 10 = 30 (Associative property of multiplication). Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56 (Distributive property). (Students need not use formal terms for these properties.)	Multiplication and Division	Properties	
3.OA.B.6: Understand properties of multiplication and the relationship between multiplication and division. Understand division as an unknown-factor problem. For example, divide 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.	Multiplication and Division	Properties	

3.OA.C.7: Multiply and divide within 100. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of one-digit numbers.	Multiplication and Division	Properties	
3.OA.D.8: Solve problems involving the four operations, and identify and explain patterns in arithmetic. Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. (This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).)	Language in Mathematics	Two-Step and Multi-Step Problems	
3.OA.D.9: Solve problems involving the four operations, and identify and explain patterns in arithmetic. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.	Language in Mathematics	Two-Step and Multi-Step Problems	
3.NBT.A.1: Use place value understanding and properties of operations to perform multi-digit arithmetic. Use place value understanding to round whole numbers to the nearest 10 or 100.			
3.NBT.A.2: Use place value understanding and properties of operations to perform multi-digit arithmetic. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. (A range of algorithms may be used.)			
3.NBT.A.3: Use place value understanding and properties of operations to perform multi-digit arithmetic. Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations. (A range of algorithms may be used.)	Multiplication and Division	Multi-Digit Multiplication	
3.NF.A.1: Develop understanding of fractions as numbers. Understand a fraction 1/b as the quantity formed by one part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Fractions as Numbers	
3.NF.A.2: Develop understanding of fractions as numbers. Understand a fraction as a number on the number line; represent fractions on a number line diagram. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Fractions as Numbers	

3.NF.A.2a: Represent a fraction 1/b on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size 1/b and that the endpoint of the part based at 0 locates the number 1/b on the number line. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Fractions as Numbers	
3.NF.A.2b: Represent a fraction a/b on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Fractions as Numbers	
3.NF.A.3: Develop understanding of fractions as numbers. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Equivalent Fractions	
3.NF.A.3a: Understand two fractions as equivalent (equal) if they are the same size or the same point on a number line. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Equivalent Fractions	
3.NF.A.3b: Recognize and generate simple equivalent fractions (e.g., 1/2 = 2/4, 4/6 = 2/3). Explain why the fractions are equivalent (e.g., by using a visual fraction model). (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Equivalent Fractions	
3.NF.A.3c: Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram. (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Equivalent Fractions	
3.NF.A.3d: Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that valid comparisons rely on the two fractions referring to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions (e.g., by using a visual fraction model). (Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.)	Fractions	Equivalent Fractions	
3.MD.A.1: Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes (e.g., by representing the problem on a number line diagram).			
3.MD.A.2: Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l) (excludes compound units such as cm^3 and finding the geometric volume of a container). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units (e.g., by using drawings (such as a	Language in Mathematics	Multiplication and Division Problem Types	

beaker with a measurement scale) to represent the problem) (excludes multiplicative comparison problems (problems involving notions of "times as much")).			
3.MD.B.3: Represent and interpret data. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. For example, draw a bar graph in which each square in the bar graph might represent 5 pets.	Language in Mathematics, Multiplication and Division	Addition and Subtraction Problem Types, Properties of Operations	
3.MD.B.4: Represent and interpret data. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot where the horizontal scale is marked off in appropriate units — whole numbers, halves, or quarters.	Fractions	Fractions as Numbers	
3.MD.C.5: Geometric measurement: understand concepts of area and relate area to multiplication and to addition. Recognize area as an attribute of plane figures and understand concepts of area measurement. a. A square with side length one unit, called "a unit square," is said to have "one square unit" of area and can be used to measure area. b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.	Multiplication and Division	Properties	
3.MD.C.6: Geometric measurement: understand concepts of area and relate area to multiplication and to addition. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).	Multiplication and Division	Properties	
3.MD.C.7: Geometric measurement: understand concepts of area and relate area to multiplication and to addition. Relate area to the operations of multiplication and addition.	Multiplication and Division	Properties	
3.MD.C.7a: Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.	Multiplication and Division	Properties	
3.MD.C.7b: Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.	Multiplication and Division	Properties	
3.MD.C.7c: Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and b + c is the sum of a × b and a × c. Use area models to represent the distributive property in mathematical reasoning.	Multiplication and Division	Properties	
3.MD.C.7d: Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.			

3.MD.D.8: Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different area or with the same area and different perimeter.			
3.G.A.1: Reason with shapes and their attributes. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.			
3.G.A.2: Reason with shapes and their attributes. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.	Fractions	Fractions as Numbers	

Grade 4 Standards	Content Focus Area	Module	Priority
4.OA.A.1: Use the four operations with whole numbers to solve problems. Interpret a multiplication equation as a comparison (e.g., interpret 35 = 5 x 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5). Represent verbal statements of multiplicative comparisons as multiplication equations.	Language in Mathematics	Multiplication and Division Problem Types	
4.OA.A.2: Use the four operations with whole numbers to solve problems. Multiply or divide to solve word problems involving multiplicative comparison (e.g., by using drawings and equations with a symbol for the unknown number to represent the problem), distinguishing multiplicative comparison from additive comparison.	Language in Mathematics	Two-Step and Multi-Step Problems	
4.OA.A.3: Use the four operations with whole numbers to solve problems. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.	Language in Mathematics	Two-Step and Multi-Step Problems	
4.OA.B.4: Gain familiarity with factors and multiples. Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.			
4.OA.C.5: Generate and analyze patterns. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule “Add 3” and the starting number 1,			

generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.			
4.NBT.A.1: Generalize place value understanding for multi-digit whole numbers. Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division. (Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.)	Multiplication and Division	Multi-Digit Multiplication, Multi-Digit Division	
4.NBT.A.2: Generalize place value understanding for multi-digit whole numbers. Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons. (Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.)	Multiplication and Division	Multi-Digit Multiplication, Multi-Digit Division	
4.NBT.A.3: Generalize place value understanding for multi-digit whole numbers. Use place value understanding to round multi-digit whole numbers to any place. (Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.)			
4.NBT.B.4: Use place value understanding and properties of operations to perform multi-digit arithmetic. Fluently add and subtract multi-digit whole numbers using the standard algorithm. (Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000. A range of algorithms may be used.)			
4.NBT.B.5: Use place value understanding and properties of operations to perform multi-digit arithmetic. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. (Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000. A range of algorithms may be used.)	Multiplication and Division	Multi-Digit Multiplication	
4.NBT.B.6: Use place value understanding and properties of operations to perform multi-digit arithmetic. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. (Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000. A range of algorithms may be used.)	Multiplication and Division	Multi-Digit Division	
4.NF.A.1: Extend understanding of fraction equivalence and ordering. Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two	Fractions	Equivalent Fractions	

fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)			
4.NF.A.2: Extend understanding of fraction equivalence and ordering. Compare two fractions with different numerators and different denominators (e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as ½). Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions (e.g., by using a visual fraction model). (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)	Fractions	Equivalent Fractions	
4.NF.B.3: Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. Understand a fraction a/b with a > 1 as a sum of fractions 1/b. (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)	Fractions	Fractions as Numbers, Addition and Subtraction of Fractions	
4.NF.B.3a: Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.	Fractions	Addition and Subtraction of Fractions	
4.NF.B.3b: Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions (e.g., by using a visual fraction model). Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8.	Fractions	Addition and Subtraction of Fractions	
4.NF.B.3c: Add and subtract mixed numbers with like denominators (e.g., by replacing each mixed number with an equivalent fraction and/or by using properties of operations and the relationship between addition and subtraction).	Fractions	Addition and Subtraction of Fractions	
4.NF.B.3d: Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators (e.g., by using visual fraction models and equations to represent the problem).	Fractions, Language in Mathematics	Addition and Subtraction of Fractions, Addition and Subtraction Problem Types	
4.NF.C.4: Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number. (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)	Fractions, Multiplication and Division	Addition and Subtraction of Fractions,	

		Multiplication and Division	
4.NF.C.4a: Understand a fraction a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4).	Fractions, Multiplication and Division	Addition and Subtraction of Fractions, Multiplication and Division	
4.NF.C.4b: Understand a multiple of a/b as a multiple of 1/b, and use this understanding to multiply a fraction by a whole number. For example, use a visual fraction model to express 3 × (2/5) as 6 × (1/5), recognizing this product as 6/5. (In general, n × (a/b) = (n × a)/b.)	Fractions, Multiplication and Division	Addition and Subtraction of Fractions, Multiplication and Division	
4.NF.C.4c: Solve word problems involving multiplication of a fraction by a whole number (e.g., by using visual fraction models and equations to represent the problem). For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?	Fractions, Language in Mathematics, Multiplication and Division	Addition and Subtraction of Fractions, Multiplication and Division, Addition and Subtraction Problem Types, Multiplication and Division Problem Types	
4.NF.D.5: Understand decimal notation for fractions, and compare decimal fractions. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. For example, express 3/10 as 30/100 and add 3/10 + 4/100 = 34/100. (Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.) (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)	Fractions	Equivalent Fractions, Addition and Subtraction of Fractions	
4.NF.D.6: Understand decimal notation for fractions, and compare decimal fractions. Use decimal notation for fractions with denominators 10 or 100. For example, rewrite 0.62 as 62/100; describe a length as 0.62 meters; locate 0.62 on a number line diagram. (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)	Fractions	Equivalent Fractions	

4.NF.D.7: Understand decimal notation for fractions, and compare decimal fractions. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions (e.g., by using a visual model). (Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)	Fractions	Equivalent Fractions	
4.MD.A.1: Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example: Know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), …			
4.MD.A.2: Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.	Fractions, Language in Mathematics	Two-Step and Multi-Step Problems, Addition and Subtraction of Fractions	
4.MD.A.3: Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit. Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length by viewing the area formula as a multiplication equation with an unknown factor.			
4.MD.B.4: Represent and interpret data. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.	Fractions	Addition and Subtraction of Fractions	
4.MD.C.5: Geometric measurement: understand concepts of angle and measure angles. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement: a. An angle is measured with reference to a circle with its center at the common endpoint of the rays by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle" and can be used to measure angles. b. An angle that turns through n 1-degree angles is said to have an angle measure of n degrees.			

4.MD.C.6: Geometric measurement: understand concepts of angle and measure angles. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.			
4.MD.C.7: Geometric measurement: understand concepts of angle and measure angles. Recognize angle measures as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems (e.g., by using an equation with a symbol for the unknown angle measure).	Language in Mathematics	Addition and Subtraction Problem Types	
4.G.A.1: Draw and identify lines and angles, and classify shapes by properties of their lines and angles. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.			
4.G.A.2: Draw and identify lines and angles, and classify shapes by properties of their lines and angles. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.			
4.G.A.3: Draw and identify lines and angles, and classify shapes by properties of their lines and angles. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.			
Grade 5 Standards	**Content Focus Area**	**Module**	**Priority**
5.OA.A.1: Write and interpret numerical expressions. Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.	Multiplication and Division	Properties	
5.OA.A.2: Write and interpret numerical expressions. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. For example, express the calculation "add 8 and 7, then multiply by 2" as 2 × (8 + 7). Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921 without having to calculate the indicated sum or product.	Multiplication and Division	Properties	
5.OA.B.3: Analyze patterns and relationships. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.			

5.NBT.A.1: Understand the place value system. Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.	Multiplication and Division	Multi-Digit Multiplication, Multi-Digit Division	
5.NBT.A.2: Understand the place value system. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole number exponents to denote powers of 10.	Multiplication and Division	Multi-Digit Multiplication, Multi-Digit Division	
5.NBT.A.3: Understand the place value system. Read, write, and compare decimals to thousandths.			
5.NBT.A.3a: Read and write decimals to thousandths using base-ten numerals, number names, and expanded form (e.g., 347.392 = 3 × 100 + 4 × 10 + 7 × 1 + 3 × (1/10) + 9 × (1/100) + 2 × (1/1000)).	Multiplication and Division	Multi-Digit Multiplication, Multi-Digit Division	
5.NBT.A.3b: Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.			
5.NBT.A.4: Understand the place value system. Use place value understanding to round decimals to any place.			
5.NBT.B.5: Perform operations with multi-digit whole numbers and with decimals to hundredths. Fluently multiply multi-digit whole numbers using the standard algorithm.	Multiplication and Division	Multi-Digit Multiplication	
5.NBT.B.6: Perform operations with multi-digit whole numbers and with decimals to hundredths. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	Multiplication and Division	Multi-Digit Division	
5.NBT.B.7: Perform operations with multi-digit whole numbers and with decimals to hundredths. Add, subtract, multiply, and divide decimals to hundredths using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.	Multiplication and Division	Multi-Digit Multiplication, Multi-Digit Division	
5.NF.A.1: Use equivalent fractions as a strategy to add and subtract fractions. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)	Fractions	Equivalent Fractions, Addition and Subtraction of Fractions	

5.NF.A.2: Use equivalent fractions as a strategy to add and subtract fractions. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators (e.g., by using visual fraction models or equations to represent the problem). Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result 2/5 + 1/2 = 3/7 by observing that 3/7 < 1/2.	Fractions, Language in Mathematics	Equivalent Fractions, Addition and Subtraction of Fractions, Addition and Subtraction Problem Types	
5.NF.B.3: Apply and extend previous understandings of multiplication and division to multiply and divide fractions. Interpret a fraction as division of the numerator by the denominator (a/b = a ÷ b). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers (e.g., by using visual fraction models or equations to represent the problem). For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3 and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?	Fractions, Language in Mathematics	Multiplication and Division of Fractions, Multiplication and Division Problem Types	
5.NF.B.4: Apply and extend previous understandings of multiplication and division to multiply and divide fractions. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.	Fractions	Multiplication and Division of Fractions	
5.NF.C.4a: Interpret the product (a/b) × q as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations a × q ÷ b. For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.)	Multiplication and Division, Fractions	Multi-Digit Multiplication, Multiplication and Division of Fractions	
5.NF.C.4b: Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.	Multiplication and Division, Fractions	Multi-Digit Multiplication, Multiplication and Division of Fractions	
5.NF.C.5: Apply and extend previous understandings of multiplication and division to multiply and divide fractions. Interpret multiplication as scaling (resizing) by: a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication. b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a	Fractions	Multiplication and Division of Fractions	

given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence a/b = (n×a) / (n×b) to the effect of multiplying a/b by 1.			
5.NF.C.6: Apply and extend previous understandings of multiplication and division to multiply and divide fractions. Solve real world problems involving multiplication of fractions and mixed numbers (e.g., by using visual fraction models or equations to represent the problem).	Fractions, Language in Mathematics	Multiplication and Division of Fractions, Multiplication and Division Problem Types	
5.NF.C.7: Apply and extend previous understandings of multiplication and division to multiply and divide fractions. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. (Students able to multiply fractions in general can develop strategies to divide fractions in general by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.)	Fractions	Multiplication and Division of Fractions	
5.NF.C.7a: Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for (1/3) ÷ 4 and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3.	Fractions	Multiplication and Division of Fractions	
5.NF.C.7b: Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5) and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4.	Fractions	Multiplication and Division of Fractions	
5.NF.C.7c: Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions (e.g., by using visual fraction models and equations to represent the problem). For example, how much chocolate will each person get if three people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?	Fractions, Language in Mathematics	Multiplication and Division of Fractions, Multiplication and Division Problem Types	
5.MD.A.1: Convert like measurement units within a given measurement system. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step real world problems.			
5.MD.B.2: Represent and interpret data. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.	Fractions	Addition and Subtraction of Fractions	

5.MD.C.3: Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition. Recognize volume as an attribute of solid figures and understand concepts of volume measurement. a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume and can be used to measure volume. b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.			
5.MD.C.4: Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition. Measure volumes by counting unit cubes using cubic cm, cubic in, cubic ft, and improvised units.			
5.MD.C.5: Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition. Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.			
5.MD.C.5a: Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths equivalently by multiplying the height by the area of the base. Represent three-fold whole-number products as volumes (e.g., to represent the associative property of multiplication).	Multiplication and Division	Properties	
5.MD.C.5b: Apply the formulas V =(l)(w)(h) and V = (b)(h) for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.			
5.MD.C.5c: Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.			
5.G.CA1: Graph points on the coordinate plane to solve real-world and mathematical problems. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis and the second number indicates how far to travel in the direction of the second axis with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).			
5.G.A.2: Graph points on the coordinate plane to solve real-world and mathematical problems. Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.			

5.G.B.3: Classify two-dimensional figures into categories based on their properties. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.			
5.G.B.4: Classify two-dimensional figures into categories based on their properties. Classify two-dimensional figures in a hierarchy based on properties.			